Theorems and Proof Outlines

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Cauchy Theorem

Let $f \in \mathcal{H}(D)$ and $\gamma \in D$ be a simple closed curve. Then

$$\int_\gamma f(z)dz = 0.$$

For simplicity, assume $f'(z) \in C(D)$.

1. $\int f(z)dz = \int udx - vdy + i\int vdx + udy$
2. **Green's Theorem**: $\int udx + vdy = \int\int (v_x - u_y)dxdy$
3. The integral $= 0$ by Cauchy-Riemann equations.

The Fundamental Theorem of Contour Integration

Let D be a simple connected region, $z_0 \in D$, $f \in \mathcal{H}(D)$. Then the function

$$\Phi(z) = \int_{z_0}^{z} f(\zeta)d\zeta, \quad z \in D$$

is analytic in D and $\Phi'(z) = f(z)$.

Because f is holomorphic, its integral is path-independent so Φ is well-defined.

1. $\frac{\Phi(z+\triangle z)-\Phi(z)}{\triangle z} = \frac{1}{\triangle z}\left(\int_z^{z+\triangle z}(f(\zeta)-f(z))d\zeta + f(z)\int_z^{z+\triangle z} d\zeta\right)$
2. For the first term: $\int_z^{z+\triangle z}|f(\zeta)-f(z)|d\zeta \leq \triangle z \sup_{\zeta\in[z,z+\triangle z]}|f(\zeta)-f(z)|$
3. First term goes to zero as $\triangle z \to 0$. Second term goes to $f(z)$.

Cauchy's Integral Formula

$f \in \mathcal{H}(D) \cap C(\bar{D})$. Then for $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

1. $\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}d\zeta = \frac{1}{2\pi i}\left(\int_{\partial D}\frac{f(\zeta)-f(z)}{\zeta-z}d\zeta + f(z)\int_{\partial D}\frac{1}{\zeta-z}d\zeta\right)$
2. First term goes to zero by deforming contour to circle centered at z with radius ϵ and using continuity. Second term is $f(z)$ by evaluating with $\zeta = z + \epsilon e^{i\theta}$.

Generalized Cauchy Integral Formula

$f \in \mathcal{H}(D) \cap C(\bar{D})$. Then for $z \in D$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

1. $\frac{f(z+\triangle z) - f(z)}{\triangle z} = \frac{1}{2\pi i \triangle z} \left(\int \frac{f(\zeta)}{\zeta - (z+\triangle z)} - \frac{f(\zeta)}{\zeta - z} d\zeta \right)$
2. Combine into one fraction, then as $\triangle z \to 0$ we recover Cauchy integral formula for first derivative.

Cauchy Estimates

Let $D = \{|z - z_0| \leq R\}$ and suppose $f \in \mathcal{H}(D) \cap C(\bar{D})$. Then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}$$

where $M = \sup_{|z-z_0|=R} |f(z)|$.

1. Use Cauchy integral formula.

Mean-value property

Let $D = \{|z - z_0| \leq R\}$ and suppose $f \in \mathcal{H}(D) \cap C(\bar{D})$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta.$$

1. Use Cauchy integral formula.

Liouville's Theorem

Entire bounded functions are constant.

1. Use Cauchy estimate.

Morera's Theorem

Let $f \in C(D)$ and suppose $\int_\gamma f(z)dz = 0$ for all simple closed curves $\gamma \in D$. Then f is analytic in D.

1. Let $F(z) = \int_{z_0}^{z} f(\zeta) d\zeta$.
2. Show that $F'(z) = f(z)$.
3. $F \in C^1(D) \implies F \in \mathcal{H}(D)$ and therefore f is analytic.

Maximum Modulus Principle

Let f be a non-constant analytic function in D. Let $M = \sup_{z\in D}$ then no $z_0 \in D$ can attain $|f(z_0)| = M$.

1. Suppose there exists $z_0 \in D$ such that $|f(z_0)| = M$.
2. Use **mean-value property** to assert that $|f(z_0 + \delta e^{i\theta})| = M$ for all $\theta \in [0, 2\pi]$ and for some $\delta > 0$.
3. Move these circles around to cover D and conclude that f must be constant.

Uniform convergence of continuous functions.

Let $f_k(z) \in C(D)$ and suppose

$$\sum_{k=1}^{n} f_k(z) \rightrightarrows s(z)$$

on D, then $s(z) \in C(D)$.

1. Take N sufficiently large such that $|s_N(z) - s(z)| < \epsilon/3$.
2. Let $\delta > 0$ be such that if $|z - z_0| < \delta$ then $|s_N(z) - s_N(z_0)| < \epsilon/3$.
3. $|s(z) - s(z_0)| \le |s(z) - s_N(z)| + |s_N(z) - s_N(z_0)| + |s_N(z_0) - s(z_0)| < \epsilon$.

Corollary: integrating uniformly convergent series.

$$\sum_{k=1}^{\infty} \int_\Gamma f_k(\zeta) d\zeta = \int_\Gamma s(\zeta) d\zeta, \quad \Gamma \in D.$$

1. $\left|\int_\Gamma s(\zeta)d\zeta - \sum_{k=1}^{n} \int_\Gamma f_k(\zeta)d\zeta\right| \le \int_\Gamma |s(\zeta) - \sum f_k(\zeta)| d\zeta < \epsilon\ell$ where ℓ is the length of Γ.

Differentiating uniformly convergent series.

Let $f_k(z) \in \mathcal{H}(D)$ and suppose

$$\sum_{k=1}^{n} f_k(z) \rightrightarrows s(z)$$

on D. Then $s(z) \in \mathcal{H}(D)$ and

$$s^{(n)}(z) = \sum_{k=1}^{\infty} f_k^{(n)}(z).$$

1. **Was not in class this day.**

Isolated Zero Theorem

A non-constant analytic function has isolated zeros.

1. If f is not identically zero and has a zero of multiplicity n at $z = a$, then $f(z) = c_n(z-a)^n + c_{n+1}(z-a)^{n+1} + \cdots = (z-a)^n(c_n + c_{n+1}(z-a) + \cdots)$.

2. The function in the parentheses is not zero at a because $c_n \neq 0$ and it is continuous because f is analytic therefore there exists a disc around a for which the function is not zero, so the zeros are isolated.

Uniqueness Theorem

If $f_1, f_2 \in \mathcal{H}(D)$ and $f_1(z_k) = f_2(z_k)$ for some sequence $\{z_k\} \to a \in D$ where $z_k \in D$ then $f_1 \equiv f_2$ in D.

1. $g := f_1 - f_2 \in \mathcal{H}(D)$

2. Apply isolated zeros property to g.

Laurent Series

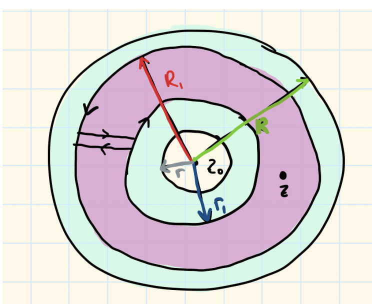
If $f \in \mathcal{H}(\Omega)$ where $\Omega = \{0 \leq r < |z - z_0| < R\}$ then

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

where

$$c_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$

for $\Gamma \in \Omega$.



1. Let z be such that $r < r_1 < |z - z_0| < R_1 < R$.

2. By Cauchy integral formula, $f(z) = \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(\zeta)}{(\zeta - z)} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{(\zeta - z)} d\zeta$.

3. Rewrite
$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \left(\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \right)$$

4. For the first integral, $\zeta \in C_{R_1}$, so $|z - z_0| < |\zeta - z_0|$ so
$$= \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}$$

5. Because of uniform convergence, we can swap the sum and integral to get

$$\frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(\zeta)}{(\zeta - z)}d\zeta = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_{R_1}}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta\right)(z - z_0)^k$$

6. Do a similar thing for the integral over C_{r_1}.

Residue Formula

Let f be holomorphic in an open set containing a circle C and its interior, except at a pole z_0 inside of C. Then

$$\int_C f(z)dz = 2\pi i \text{Res}(f, z_0)$$

1. We can use a keyhole contour to get $\int_C f(z)dz = \int_{C_\epsilon(z_0)} f(z)dz$.
2. From **Laurent series**, we have that near z_0 we can write $f(z) = \frac{c_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$ where G is holomorphic.
3. We can integrate each of these terms and see that they are all zero except for one being $2\pi i a_{-1}$.

Corollary: Cauchy's Residue Theorem

Let $f \in \mathcal{H}(D\backslash\{z_1, \cdots, z_n\}) \cap C(\bar{D}\backslash\{z_1, \cdots, z_n\})$. Then

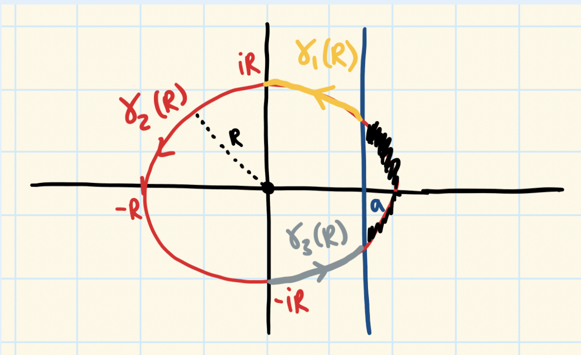
$$\int_{\partial D} f(z)dz = 2\pi i \sum_{k=1}^{n}\text{Res}(f, z_k)$$

1. Do multiple keyholes.

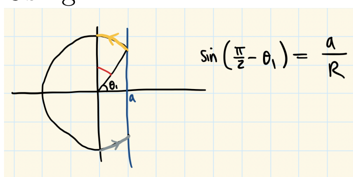
Jordan's Lemma

Let $f(z)$ be analytic except for isolated singularities with $\text{Re}z_k > a > 0$ and $f(\infty) = 0$. Let $\gamma(R) = \gamma_1(R) \cup \gamma_2(R) \cup \gamma_3(R)$ be as shown. Then,

$$\lim_{R\to\infty}\int_{\gamma(R)} e^{tz}f(z)dz = 0, \ \ t > 0.$$



1. Let $M(R) = \max_{z \in \gamma(R)} |f(z)|$.
2. $\left| \int_\gamma e^{tz} f(z) dz \right| \leq M(R) \int_\gamma |e^{tz}||dz|$
3. Using

   we have

$$I_1 = M(R) \int_{\gamma_1} |e^{tx}||dz| \leq M(R) e^{ta} \int_{\theta_1}^{\pi/2} R d\theta$$
$$= M(R) e^{ta} R \arcsin(a/R) = M(R) e^{ta} R \left(\frac{a}{R} + O(\frac{a^3}{R^3}) \right) \to 0.$$

4. Similarly $I_3 \to 0$.
5. $I_2(R) = M(R) \int_{\pi/2}^{3\pi/2} e^{tR\cos\theta} R d\theta$.

Corollary

$$I(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{tz} f(z) dz = \sum_{k=1}^{n} \mathrm{Res}(e^{tz} f(z), z_k)$$

Logarithmic Residue Lemma

(a) Let z_0 be a zero of multiplicity n of a meromorphic function $f(z)$ in a region D, that is, $f \in \mathcal{M}(D)$. Then,

$$\mathrm{Res}\left(\frac{f'(z)}{f(z)}, z_0 \right) = n = c_{-1}$$

(b) Let z_0 be a pole of order p of $f \in \mathcal{M}(D)$. Then,

$$\mathrm{Res}\left(\frac{f'(z)}{f(z)}, z_0 \right) = -p = c_{-1}$$

1. Near z_0, $f(z) = c_n(z - z_0)^n + c_{n+1}(z - z_0)^{n+1} + \cdots$
2. So $f'(z) = nc_n(z - z_0)^{n-1} + (n+1)c_{n+1}(z - z_0)^n + \cdots$
3. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{n + O(z - z_0)}{(z - z_0) + O(z - z_0)^2}$$

   has a pole of order 1 and has residue n.

4. For (b), use Laurent series of f.

Integral of logarithmic derivative.

Let $f \in \mathcal{M}(D)$ and let $D_1 \subset D$ not contain any poles or zeros of $f(z)$. If N is the number of zeros in D_1 and P is the number of poles, then

$$\frac{1}{2\pi i}\int_{\partial D_1}\frac{f'(\zeta)}{f(\zeta)}d\zeta = N - P.$$

1. Follows immediately from residue theorem and logarithmic residue lemma.

Argument Principle

Under the above assumptions, the difference between the number of zeros and poles is equal to the number of times the argument of $f(z)$ winds around the origin when making one traversal of ∂D_1:

$$N - P = \frac{\triangle \arg f(z)}{2\pi}.$$

1. Use that fact that the integrand is the logarithmic derivative of f:

$$N - P = \frac{1}{2\pi i}\int_{\partial D_1}\frac{f'(\zeta)}{f(\zeta)}d\zeta = \frac{1}{2\pi i}\int_{\partial D_1} d\log f(\zeta)d\zeta = \frac{1}{2\pi i}\left(\log|f(z)| + i\arg f(z)\right)\Big|_{\partial D_1}$$

Rouche's Theorem

Let $f, g \in \mathcal{H}(D) \cap C(\partial D)$ be such that $|f| > |g|$ on ∂D. Then the functions $F = f + g$ and f have the same number of zeros in D.

1. Rewrite $F(z) = f(z)\left(1 + \frac{g(z)}{f(z)}\right) =: f(z)w(z)$
2. By Argument Principle, $N_F = \frac{\triangle \arg f + \triangle \arg w}{2\pi}$.
3. $|w - 1| = |g/f| < 1$ so there is no rotation around the origin, so $\triangle \arg w = 0$.

Fundamental Theorem of Algebra

Let $P_n(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,\ a_0 \neq 0$. Then P_n has exactly n zeros in $\mathbb{C}$.

1. Let $C_R = \{|z| = R\}$ with R sufficiently large such that all zeros are enclosed.
2. $P_n(z) = f(z) + g(z)$ where $f(z) = a_0 z^n$ and $g(z) = a_1 z^{n-1} + \cdots + a_n$.
3. On C_R, $|f| = a_0 R^n$ and $|g| \leq nMR^{n-1}$ where $M = \max\{|a_1|, \cdots, |a_n|\}$.
4. Can choose R suff. large so that $|f| > |g|$ and then apply Rouche.

Proposition

Let $\{a_k\} \in \mathbb{C}\backslash\{-1\}$. Then,

$$\prod_{k=1}^{\infty}(1 + a_k) \text{ converges iff } \sum_{k=1}^{\infty}\ln(1 + a_k) \text{ converges.}$$

1. Log of products is sum of logs.

Proposition

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \quad \Longrightarrow \prod_{k=1}^{\infty}(1+a_k) \text{ converges.}$$

1. Disregarding possibly a finite number of terms, we can assume $|a_n| < 1/2$ for all n.
2. For $|z| < 1$, we have $1 + z = e^{\ln(1+z)}$.
3. Rewrite the infinite product

$$\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\ln(1+a_n)} = \exp\left(\sum_{n=1}^{N} \ln(1+a_n)\right)$$

4. By the power series expansion of $\log(1+z)$ we see that $|\log(1+z)| \le 2|z|$ if $|z| < 1/2$.

Proposition

An absolutely convergent product is convergent, i.e.

$$\prod_{k=1}^{\infty}(1+|a_k|) \text{ converges} \quad \Longrightarrow \prod_{k=1}^{\infty}(1+a_k) \text{ converges.}$$

Theorem

Let $\{f_k(z)\} \in \mathcal{H}(D)$ and suppose $\sum_{k=1}^{\infty} |f_k(z)|$ converges uniformly on compacta. Then,

$$\prod_{k=1}^{\infty}(1+f_k(z))$$

converges uniformly on compacta and represents an analytic function in D.

Weierstrass Product Theorem

Suppose $\{a_k\} \to \infty$. Then there exists an analytic function $f(z)$ such that $f(z) = 0$ iff $z = a_k$ for $k = 1, 2, \cdots$.

1. For each k, define

$$E_k(z) = \exp\left(\sum_{j=1}^{k} \frac{z^j}{j a_k^j}\right)$$

2. Suppose $|z| < M$. Then since $a_k \to \infty$, for sufficiently large k we have $|a_k| > 2|z|$.
3. Use $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

4. Then we can bound

$$\begin{aligned}
\left|\log((1-\frac{z}{a_k})E_k(z))\right| &= |\log(1-z/a_k)+\log(E_k(z))| \\
&= \left|-\frac{z}{a_k}-\frac{z^2}{2a_k^2}-\cdots+\sum_{j=1}^{k}\frac{z^j}{ja_k^j}\right| \\
&= \left|\sum_{j=k+1}^{\infty}\frac{z^j}{ja_k^j}\right| \\
&\le \sum_{j=k+1}^{\infty}\left|\frac{z}{a_k}\right|^j \\
&\le \left|\frac{z}{a_k}\right|^k \\
&\le \frac{1}{2^k}
\end{aligned}$$

5. Therefore by Weierstrass M-test, $\sum \log((1-\frac{z}{a_k})E_k(z))$ converges uniformly on compacta, so $\prod(1-\frac{z}{a_k})E_k(z)$ converges uniformly on compacta and is entire.

Mittag-Leffler's Expansion Theorem

Assume that the only singularities of $f(z)\in\mathcal{H}(\mathbb{C}\backslash\{a_1,a_2,\cdots\})$ are simple poles where $|a_k|\le|a_{k+1}|$ and $a_k\neq 0$. Let $b_k=\text{Res}(f(z),a_k)$ and C_N be circles of radius $R_N\to\infty$ that do not pass through any poles and on which $|f(z)|<M$, where M is independent of N. Then

$$f(z)=f(0)+\sum_{k=1}^{\infty}b_k\left(\frac{1}{z-a_k}+\frac{1}{a_k}\right)$$

1. Suppose ζ is not a pole. Then $\text{Res}(\frac{f(z)}{z-\zeta},a_k)=\frac{b_k}{a_k-\zeta}$ and $\text{Res}(\frac{f(z)}{z-\zeta},a_k)=f(\zeta)$.
2. By Residue Theorem,
$$\frac{1}{2\pi i}\int_{C_N}\frac{f(w)}{w-\zeta}dw=f(\zeta)+\sum_{k=1}^{n}\frac{b_k}{a_k-\zeta}$$
3. In particular, because f holomorphic at $\zeta=0$, we have
$$\frac{1}{2\pi i}\int_{C_N}\frac{f(w)}{w}dw=f(0)+\sum_{k=1}^{n}\frac{b_k}{a_k}$$
4. (2) - (3) gives
$$f(\zeta)-f(0)=\sum_{k=1}^{n}\frac{b_k}{a_k}-\frac{b_k}{a_k-\zeta}+\frac{1}{2\pi i}\int_{C_N}\left(\frac{1}{w-\zeta}-\frac{1}{w}\right)f(w)dw$$
5. Combine the fractions in the integrand and show that the integral goes to zero as $N\to\infty$.

Local Property

If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then there is an open disk D centered at z_0 such that f is one-to-one on D.

1. By continuity of f' at z_0 there exists an open disk $D(z_0)$ such that $|f'(z) - f'(z_0)| \leq \frac{|f'(z_0)|}{2}$ for all $z \in D(z_0)$.

2. Let $\gamma \in D(z_0)$ be a path joining $z_1 \neq z_2$. Then

$$\begin{aligned}
|f(z_1) - f(z_2)| &= |\int_\gamma f'(z)dz| \\
&= |\int_\gamma f'(z) - f'(z_0) + f'(z_0)dz| \\
&\geq |f'(z_0)||z_2 - z_1| - |\int_\gamma f'(z) - f'(z_0)dz| \\
&\geq |f'(z_0)||z_2 - z_1| - \int_\gamma |f'(z) - f'(z_0)|dz \\
&\geq |f'(z_0)||z_2 - z_1| - |f'(z_0)/2||z_2 - z_1| \\
&> 0.
\end{aligned}$$

Conformality.

An analytic function $f(z)$ is conformal at every point z_0 for which $f'(z_0) \neq 0$.

1. Consider a curve $\gamma = \{z(t) : a \leq t \leq b\}$. Then $z'(t_0)$ is tangent at $z_0 = z(t_0)$.

2. Consider its image $w(t) = f(z(t))$. By **chain rule**, $w'(t_0) = f'(z(t_0))z'(t_0)$.

3. So $\arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0)$, i.e. all curves through z_0 are rotated by the same angle $\arg f'(z_0)$.

Open Mapping Theorem

If D is an open connected region, $f \in \mathcal{H}(D)$, and f is not constant, then $f(D)$ is open.

1. Let $z_0 \in D$ be such that $f(z_0) = 0$ (if none exists, shift f). Consider a disk around z_0, $D_\epsilon(z_0)$.

2. By **isolated zeros**, there exists $C_r(z_0)$ $(r < \epsilon)$ such that $f(z) \neq 0$ on C_r.

3. Let $m := \min_{z \in C_r} |f(z)| > 0$ then let $|\alpha| < m$. By **Rouche's theorem**, since f has a zero in C_r, so does $f - \alpha$ so $f(\zeta) = \alpha$ for some ζ in C_r. Therefore, the image of f in C_r contains the disk $|w| < m$.

4. The same thing can be done at any point z_0 by shifting f.

Continuity Principle

Let D_1, D_2 that do not intersect but share a smooth boundary γ. Suppose $f_i \in \mathcal{H}(D_i) \cap C(\partial D_i)$ for $i = 1, 2$ and $f_1(z) = f_2(z)$ on γ. Then,

$$f(z) = \begin{cases} f_1(z) & z \in D_1 \cup \gamma \\ f_2(z) & z \in D_2 \end{cases}$$

is analytic in $D_1 \cup \gamma \cup D_2$.

This proof is inspired by the proof of Schwarz Reflection Principle in Complex Analysis (Stein and Shakarchi).

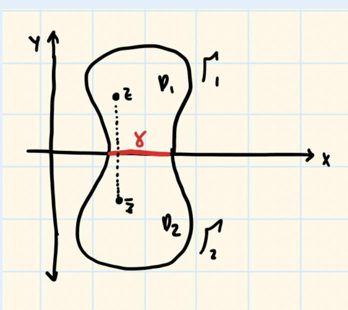
1. Of course f is holomorphic in D_1 and D_2, so we need to show that it is holomorphic on γ.
2. Because it is continuous, by **Morera's theorem**, we just need to show that integrals over closed contours that cross γ are zero.
3. Consider $z_0 \in \gamma$ and a disk $D_\delta(z_0)$. Consider $A_1 = \partial(D_1 \cap D_\delta)$ and $A_2 = \partial(D_2 \cap D_\delta)$.
4. Let A_1^ϵ be A_1 moved a distance ϵ from γ (further into D_1). Then clearly $\int_{A_1^\epsilon} f(z)dz = 0$ so by continuity, letting $\epsilon \to 0$ gives $\int_{A_1} f = 0$. Same for A_2. Therefore $\int_{C_\delta} f(z)dz = 0$.

Schwarz's Reflection Principle

Let D_1 be a region as shown below and let D_2 be the reflection of D_1 over the real axis. Suppose $f_1(z) = \mathcal{H}(D_1) \cap C(\gamma)$ and $f_1(\gamma) \subset \mathbb{R}$. Then f_1 can be analytically extended to D_2 as

$$f(z) = \begin{cases} f_1(z) & z \in D_1 \cup \gamma \\ f_2(z) & z \in D_2 \end{cases}$$

where $f_2(z) = \overline{f_1(\bar{z})}$.



1. Show that $f_2 \in \mathcal{H}(D_2)$:

$$\frac{f_2(z + \triangle z) - f_2(z)}{\triangle z} = \frac{\overline{f_1(\overline{z + \triangle z})} - \overline{f_1(\bar{z})}}{\triangle z} = \overline{\left(\frac{f_1(\overline{z + \triangle z}) - f_1(\bar{z})}{\overline{\triangle z}}\right)}$$

Therefore, $f_2'(z) = \overline{f_1'(\bar{z})}$, so f_2 is analytic in D_2.

2. Clearly $f_1 = f_2$ on γ, therefore by **continuity principle**, f is analytic.

Proposition

Let $f \in \mathcal{H}(D)$ and one-to-one on $D_1 \subset D$. Then $f'(z) \neq 0$ for all $z \in D_1$.

1. Prove the **contrapositive**. Suppose there exists $z_0 \in D_1$ such that $f'(z_0) = 0$.
2. The Taylor series of f near z_0 is $f(z) = a_0 + \phi(z) = a_0 + \sum_{k=n}^{\infty} a_k(z - z_0)^k$ where $a_n \neq 0$ and $n \geq 2$.
3. ϕ has a zero of multiplicity n at $z = z_0$, and it is analytic so it must have **isolated zeros**. Therefore for some δ, $\phi \neq 0$ on $C_\delta(z_0)$.
4. Let $m = \min_{|z-z_0|=\delta} |\phi(z)| > 0$ and let α be such that $0 < |\alpha| < m$.
5. Define $\psi = \phi - \alpha$. By **Rouche's theorem**, ψ must also have n zeros in $D_\delta(z_0)$. Therefore, $f = \alpha + a_0$ at least twice, so f is not one-to-one.

Schwarz Lemma

If f is analytic on the unit disk and satisfies

1. $f(0) = 0$
2. $|f| \leq 1$ on $|z| < 1$

then $|f(z)| \leq |z|$ on $|z| < 1$.

1. Define
$$g(z) = \begin{cases} \frac{f(z)}{z} & 0 < |z| < 1 \\ f'(0) & z = 0. \end{cases}$$
Clearly, g is analytic on $0 < |z| < 1$. It is also analytic at $z = 0$:
$$\lim_{z \to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - zf'(0)}{z^2} = \frac{f''(0)}{2}$$
by Taylor series.
2. Take $\zeta \in D$ and r such that $|\zeta| < r < 1$. By **Maximum Modulus Principle**, $|g(\zeta)| \leq \max_{|z|=r} \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$.
3. Taking $r \to 1^-$ we get $|g(\zeta)| \leq 1$ so $|f(\zeta)| \leq |\zeta|$.

Definitions.

1. An **isomorphism** is a conformal one-to-one map.
2. D_1 and D_2 that admit an isomorphism are called **isomorphic** or **conformally equivalent**.
3. An isomorphism of D onto itself is called an **automorphism**.

Proposition

Let $g : D_1 \to D_2$ be a given isomorphism and $\phi : D_2 \to D_2$ an automorphism. Then any other isomorphism of D_1 onto D_2 has the form $f = \phi \circ g$.

1. Take an arbitrary isomorphism $f : D_1 \to D_2$. Then $\phi := f \circ g^{-1}$ is an automorphism of D_2. So $f = \phi \circ g$.

Theorem

Any conformal automorphism of a canonical region (unit disk, $\mathbb{C}$, $\bar{\mathbb{C}}$) is a Mobius transformation,

$$f(z) = \frac{az+b}{cz+d}.$$

1. This was not proven in class.

Uniqueness of Riemann Mapping Theorem

If a region D is isomorphic to the unit disk Ω, then there exists a unique conformal map $f : D \to \Omega$ such that

$$f(z_0) = 0, \quad \arg f'(z_0) = \theta$$

where $z_0 \in D$ is an arbitrary point and θ is an arbitrary real.

In other words, the set of all conformal maps of D onto Ω depends on 3 real parameters.

1. This was not proven in class.

Definitions.

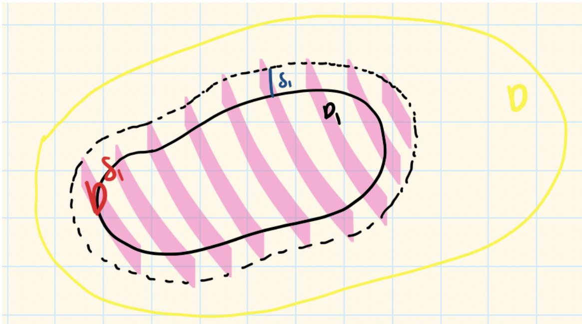
1. A family of functions $\mathcal{F}$ on D is **locally uniformly bounded** if for any $D_1 \subset D$ and for all $f \in \mathcal{F}$ there exists $M = M(D_1) > 0$ such that for all $z \in D_1$, $|f(z)| \leq M$.
2. $\mathcal{F}$ is **locally equicontinuous** if for any $D_1 \subset D$ and for all $f \in \mathcal{F}$, we have that for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, D_1)$ such that $\forall |z_1 - z_2| < \delta$, $|f(z_1) - f(z_2)| < \epsilon$.
3. $\mathcal{F}$ is **compact** if any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ has a subsequence that converges uniformly on any $D_1 \subset D$.

Lemma

If a family of functions $\mathcal{F}$ on a region D is locally uniformly bounded, then it is locally equicontinuous.

1. Let $D_1 \subset D$ and $\delta_1 = \frac{1}{2}\text{dist}(\bar{D}_1, \partial D)$. Define

$$D^{\delta_1} = \bigcup_{z_1 \in D_1} \{z \in D : |z - z_1| < \delta_1\}.$$



2. By locally uniformly bounded, there exists $M = M(D^{\delta_1})$ such that for all $f \in \mathcal{F}$, $|f(z)| \leq M$ $\forall z \in D^{\delta_1}$.

3. Let $a \in D_1$ and consider $g(\zeta) = \frac{1}{2M}(f(a + \zeta\delta_1) - f(a))$ on $|\zeta| < 1$. Since $g(0) = 0$ and $|g(\zeta)| \leq 1$, by **Schwarz Lemma**, $|g(\zeta)| \leq |\zeta|$ for all $|\zeta| < 1$.

4. Using $\zeta = \frac{1}{\delta_1}(z - a)$, (3) implies that $|f(z) - f(a)| \leq \frac{2M}{\delta_1}|z - a|$. Given $\epsilon > 0$, let $\delta = \frac{\epsilon\delta_1}{2M}$ then if $|z - a| < \delta$ we have $|f(z) - f(a)| < \epsilon$.

Montel's Theorem

If a family of analytic functions $\mathcal{F}$ on D is locally uniformly bounded, then it is compact.

Part 1: $\{f_n\} \in \mathcal{F}$ has a subsequence that converges on a dense subset of $\mathbb{E} \subset D$.

1. Let $\mathbb{E} = \{z \in D : z = x + iy, x, y \in \mathbb{Q}\}$ and choose an ordering $\mathbb{E} = \{z_m\}_{m=1}^{\infty}$.

2. By locally uniformly bounded, the sequence $\{f_n(z_1)\}$ is bounded and hence has a convergent subsequence $f_{k,1} := f_{n_k}(z_1)$.

3. $f_{n_k}(z_2)$ is bounded and must have a convergent subsequence $f_{j,2} = f_{n_{k_j}}(z_2)$.

4. Construct the sequence $f_{1,1}, f_{2,2}, f_{3,3} \cdots$ which converges at all $z \in \mathbb{E}$.

Part 2: If a sequence $\{f_n\}$ converges at every point on a dense set $\mathbb{E} \subset D$ then it converges uniformly on every compact subset $D_1 \subset D$.

1. Let $\epsilon > 0$. Choose a partitioning of D_1 into a finite number of squares $A_p, p = 1, \cdots, q$ such that if $a, b \in A_p$ then $|f_n(a) - f_n(b)| < \epsilon/3$ for all $n = 1, \cdots$ (valid by equicontinuity).

2. Because $\mathbb{E}$ is dense, there exists $z_p \in \mathbb{E}$ in each A_p.

3. Because f_n converges on $\mathbb{E}$: there exists $N \in \mathbb{N}$ such that for $n, m > N$ we have

$$|f_n(z_p) - f_m(z_p)| < \epsilon/3.$$

4. Let $z \in D_1$ and let $z_p \in \mathbb{E}$ be in the same partitioned square. Then for $n, m > N$,

$$|f_m(z) - f_n(z)| \leq |f_m(z) - f_m(z_p)| + |f_m(z_p) - f_n(z_p)| + |f_n(z_p) - f_n(z)| < \epsilon.$$

Definitions.

1. Let $\mathcal{F}$ be a family of functions on D. A functional $J : \mathcal{F} \to \mathbb{C}$ is **continuous** if for all $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ that converge uniformly on any compact subset $D_1 \subset D$,

$$\lim_{n\to\infty} J(f_n) = J(\lim_{n\to\infty} f_n).$$

2. A compact family of functions $\mathcal{F}$ is **sequentially compact** if the limit function of any sequence $\{f_n\} \in \mathcal{F}$, that converges uniformly on any compact subset $D_1 \subset D$, belongs to $\mathcal{F}$.

Lemma

Any functional J that is continuous on a sequentially compact family $\mathcal{F}$ is bounded and attains its supremum, that is $\exists f_0 \in \mathcal{F}$ such that $|J(f_0)| \geq |J(f)| \;\; \forall f \in \mathcal{F}$.

1. Denote $M = \sup_{f\in\mathcal{F}} |J(f)|$. By definition of supremum, there exists $\{f_n\} \in \mathcal{F}$ such that $|J(f_n)| \to M$ as $n \to \infty$.
2. Because $\mathcal{F}$ is sequentially compact, there is a subsequence of $\{f_n\}$, $\{f_{n_k}\} \rightrightarrows f_0 \in \mathcal{F}$.
3. By continuity of J, $|J(f_0)| = M$.

Hurwitz's Theorem

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}(D)$ be such that $f_n \rightrightarrows f \neq \text{const.}$ on all $D_1 \subset D$. Then if $f(a) = 0$ for some $a \in D$, given any disk $D_\delta(a)$ there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $f_n(z)$ vanishes at some $D_\delta(a)$.

1. By Weierstrass, $f \in \mathcal{H}(D)$. By **isolated zeros**, there exists a circle $C_\delta(a)$ on which $f \neq 0$.
2. Define $m = \min_{|z-a|=\delta} |f(z)| > 0$. By uniform convergence, there exists N such that for all $n \geq N$, $|f_n(z) - f(z)| < m$ for all $z \in C_\delta$.
3. For $n \geq N$, rewrite $f_n(z) = f_n(z) - f(z) + f(z)$. By **Rouche**, f_n and f have the same number of zeros in $D_\delta(a)$.

Corollary

Let $\{f_n\}_{n=1}^{\infty} \in D$ be a sequence of univalent (injective + analytic) functions. If $\{f_n\} \rightrightarrows f$ on any $D_1 \subset D$ then f is either constant or univalent.

1. Prove the **contrapositive**. Suppose $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$ and f not constant.
2. Consider a sequence $g_n(z) = f_n(z) - f_n(z_2)$ and a disk $D_r(z_1)$ where $0 < r < |z_1 - z_2|$. Then $g_n \rightrightarrows g(z) = f(z) - f(z_2)$ which has a zero at z_1.
3. By Hurwitz, there exists N such that for all $n \geq N$, g_n has a zero in $D_r(z_1)$, i.e. f_n is not univalent.

Riemann's Mapping Theorem

Any simply connected region D in $\mathbb{C}$ or $\bar{\mathbb{C}}$ with a boundary containing more than one point is conformally equivalent to the unit disk.