ODE Notes

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Superposition principle.

If we have a linear homogeneous ODE of the form

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_1 + x_2 \end{cases}$$

then we can solve the ODE separately for initial data $x(0) = x_1$ and $x(0) = x_2$ and add the solutions together to get a solution.

If we have an inhomogeneous ODE with zero initial data of the form

$$\begin{cases} \dot{x} = Ax + b_1 + b_2 \\ x(0) = 0 \end{cases}$$

then we can solve $\dot{x} = Ax + b_1$ and $\dot{x} = Ax + b_2$ separately and add the solutions together to get a solution.

Combining these two ideas, if we have an ODE of the form

$$\begin{cases} \dot{x} = Ax + b_1 + b_2 \\ x(0) = x_1 + x_2 \end{cases}$$

we can solve

$$\begin{cases} \dot{x} = Ax + b_1 \\ x(0) = x_1 \end{cases}$$

and

$$\begin{cases} \dot{x} = Ax + b_2 \\ x(0) = x_2 \end{cases}$$

separately and add the solutions to get a solution.

Example. Consider the IVP

$$\begin{cases} \dot{x}(t) = a(t)x(t) + b(t) \\ x(0) = x_0. \end{cases}$$

Using the superposition principle we can separately solve

$$\begin{cases} \dot{x}(t) = a(t)x(t) \\ x(0) = x_0 \end{cases}$$

and

$$\begin{cases} \dot{x}(t) = a(t)x(t) + b(t) \\ x(0) = 0. \end{cases}$$

The first equation is separable, and has solution

$$x(t) = x_0 e^{\int_0^t a(s)ds}.$$

For the second equation, we can deal with the forcing term $b(t)$ via Duhamel's principle, i.e. by thinking of it as initial data that is being added at each time. Observe that

$$g(t; s) = b(s)e^{\int_s^t a(s)ds}$$

solves

$$\dot{g}(t) = a(t)g(t), \quad g(s) = b(s)$$

therefore if we add up the g's for all the values of $0 \leq s \leq t$, we should get a solution:

$$x(t) = \int_0^t b(\tau)e^{\int_\tau^t a(s)ds} d\tau.$$

Fixed point theorems.

Let $(X, \|\cdot\|)$ be a normed vector space. A mapping $F : X \to Y$ is **continuous** if $f_n \to f \implies F(f_n) \to F(f)$. A normed space is called **complete** if every Cauchy sequence has a limit. A **Banach space** is a complete normed space.

Let I be a compact interval and consider the continuous functions $C(I)$ on it. $C(I)$ becomes a normed space if we define

$$\|x\| = \sup_{t \in I} |x(t)|.$$

Note that $C(I)$ is a Banach space. It is complete because if $x_n \to x$ then $\lim_{n\to\infty} \sup_{t\in I} |x_n - x| \to 0$ i.e. $x_n(t) \to x(t)$ uniformly and since uniform convergence preserves continuity, $x(t) \in C(I)$.

A **fixed point** of a mapping $K : C \subseteq X \to C$ is an element $x \in C$ such that $K(x) = x$. K is called a **contraction** if there is a contraction constant $\theta \in [0, 1)$ such that

$$\|K(x) - K(y)\| \leq \theta\|x - y\|.$$

We will use the following notation: $K^n(x) = K(K^{n-1}(x))$, $K^0(x) = x$.

Contraction Principle.

Let C be a nonempty closed subset of a Banach space X and let $K : C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ such that

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta}\|K(x) - \bar{x}\|$$

for $x \in C$. *So the contraction principle tells us that if K is a contraction, then we can find its unique fixed point, $\bar{x}$ s.t. $K(\bar{x}) = \bar{x}$, by repeatedly applying K to any starting point x.*

Proof:
Uniqueness: let $\bar{x}$ and $\tilde{x}$ be two fixed points of K. Then

$$\|\bar{x} - \tilde{x}\| = \|K(\bar{x}) - \tilde{x}\| \leq \frac{\theta}{1-\theta}\|K(\bar{x}) - \bar{x}\| = 0.$$

Existence: $\cdots$

The basic existence and uniqueness result.

Consider an initial value problem

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

where $f \in C(U, \mathbb{R}^n)$ where U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If we integrate the ODE with respect to t, we get the following **integral equation:**

$$x(t) = x_0 + \int_0^t f(s, x(s))dx.$$

Note that for small t, $x_0(t) = x_0$ is an approximate solution. Plug this in to get another approximate solution:

$$x_1(t) = x_0 + \int_0^t f(s, x_0(s))ds.$$

We can iterate this process to get a sequence of approximating solutions. This defines the **Picard iteration:**

$$x_m(t) = K^m(x_0)(t), \quad K(x)(t) = x_0 + \int_0^t f(s, x(s))ds.$$

Notice that if $x(t)$ solves the integral equation, then it is a FP $x = K(x)$. We can apply the **contraction principle** to get existence and uniqueness for our IVP.

Linear ODEs and their fundamental solution.

Now we restrict our conversation to linear ODEs of the form

$$\dot{x} = A(t)x + b(t).$$

Example: complex ODEs can be analyzed as 2D systems of real ODEs. Consider the ODE

$$\dot{z}(t) = iz(t)$$

which clearly has solution $z(t) = z(0)e^{it}$. Let us rewrite this as a system of equations for x and y, where $z = x + iy$. Then

$$\dot{x} + i\dot{y} = ix - y$$

gives

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x. \end{cases}$$

We can turn this into a second order ODE by combining the two equations to get

$$\ddot{x} + x = 0$$

which has solution

$$x(t) = x_0 \cos t - \dot{x}_0 \sin t.$$

Back to the general case, assume A is bounded:

$$\|A\| = \sup_x \frac{\|Ax\|}{\|x\|} \leq M.$$

Then observe that $f(t, x) = A(t)x + b(t)$ is Lipschitz in x, since

$$\|Ax + b - Ay - b\| = \|A(x - y)\| \leq \|A\|\|x - y\| \leq M\|x - y\|.$$

Therefore, we know that we have existence and uniqueness. For simplicity, suppose $b = 0$ and A is constant in time:

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0. \end{cases}$$

We can seek a solution by applying the Picard iteration:

$$x_0(t) = x_0$$

$$x_1(t) = x_0 + \int_0^t Ax_0 ds = x_0 + Ax_0 t = (I + At)x_0$$

$$x_2(t) = x_0 + \int_0^t A(I + As)x_0 ds = (I + At + \frac{A^2t^2}{2})x_0$$

$$\vdots$$

$$x_n(t) = \left(I + At + \cdots + \frac{A^n t^n}{n!}\right) x_0$$

Therefore we get that the solution is

$$x(t) = \lim_{n\to\infty} x_n(t) = \left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) x_0 = e^{At}x_0.$$

We can alternatively arrive at this solution by solving with Euler's method with time step of size $\triangle t = t/n$:

$$x_n(\triangle t) = x_0 + Ax_0\triangle t = (I + A\triangle t)x_0$$

$$x_n(2\triangle t) = x_n(\triangle t) + Ax_n(\triangle t)\triangle t = x_0 + Ax_0\triangle t + A(x_0 + Ax_0\triangle t)\triangle t = (I + A\triangle t)^2 x_0$$

$$\vdots$$

$$x_n(t) = (I + A\triangle t)^n x_0$$

Fact: $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

$$x(t) = \lim_{n\to\infty} x_n(t) = e^{At}x_0.$$

If A is diagonalizable, i.e.

$$A = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1},$$

then we can write

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} P^{-1}.$$

Non-autonomous linear ODEs.

Now we let $A = A(t)$. Consider the 1D case,

$$\begin{cases} \dot{x} = a(t)x \\ x(0) = x_0, \end{cases}$$

where we had solution

$$x(t) = x_0 e^{\int_0^t a(s)ds}.$$

Because of this, we might be tempted to postulate that in the n-dimensional case,

$$x(t) = e^{\int_0^t A(s)ds} x_0,$$

but this won't work. Heuristically, this integral is a sum of $A(t)$'s and it doesn't care about order but of course the order in which $A(t)$ takes different values in time should matter. We will not be able to find a closed form solution, but we can make some general statements. We seek a **fundamental solution** $\Pi(t)$ that solves

$$\begin{cases} \dot{\Pi}(t) = A(t)\Pi(t) \\ \Pi(0) = I. \end{cases}$$

We call this the fundamental solution because

$$x(t) := \Pi(t)x_0$$

solves

$$\begin{cases} \dot{x}(t) = A(t)x(t) \\ x(0) = x_0. \end{cases}$$

In class, we went through a long-ish proof that Π stays non-singular (invertible) for all time. It was important that $\Pi(t)$ be invertible because this means that for any initial condition x_0 there is a unique solution $x(t) = \Pi^{-1}x_0$.

Now back to the forced problem,

$$\dot{x} = A(t)x + b(t), \quad x(0) = x_0,$$

equipped with the fundamental solution, we can use Duhamel's principle to write down the solution. We have that

$$g(t; s) := \Pi(t - s)b(s)$$

solves

$$\dot{g} = A(t)g, \quad g(s;s) = b(s)$$

and thus our solution is

$$x(t) = \Pi(t)x_0 + \int_0^t \Pi(t-s)b(s)ds.$$

General solutions.

If we have an ODE of the form

$$\dot{x}(t) = A(t)x(t) + b(t)$$

and we can guess a particular solution $x_p(t)$ then our solution is

$$x(t) = x_h(t) + x_p(t)$$

Jordan form.

Commuting matrices.

If $[A, B] = AB - BA = 0$, then

$$e^{A+B} = e^A e^B.$$

A Jordan block can be decomposed:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

These two matrices commute (the identity commutes with everything). Therefore, when we exponentiate a Jordan block, we can write it as the product of exponentials:

$$\exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \exp(\lambda I)\exp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

This matrix with ones on the off-diagonals can only be raised to a finite number of powers before becoming the zero matrix. Therefore its exponential form can be written as a finite polynomial of matrices. At this point, we have solved the general linear system

$$\dot{x} = Ax$$

for any A (since every square matrix has a Jordan decomposition).

Boundary value problems for PDEs.

Consider the wave equation in 1D:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,t) = u(\pi, t) = 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_{t0}(x) \end{cases}$$

We will use **separation of variables** and guess a solution of the form

$$u(x,t) = X(x)T(t).$$

Plugging this in gives

$$\begin{aligned} T''(t)X(x) &= X''(x)T(t) \\ \implies \frac{T''(t)}{T(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

and this can only be true if both sides are equal to a constant $-\lambda^2$. We thus obtain the following 2 ODEs:

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0, \quad X(0) = X(1) = 0 \\ T''(t) + \lambda^2 T(t) = 0. \end{cases}$$

Let us define the operator

$$L(X(x)) = X''(x)$$

on the space of functions with homogeneous boundary conditions $X(0) = X(\pi) = 0$. Then L is **self-adjoint** with respect to the L2 inner product, i.e.

$$\int_0^\pi L(\varphi(x))\psi(x)dx = \int_0^\pi \varphi(x)L(\psi(x))dx.$$

Self-adjoint operators have real eigenvalues, so this justifies making our constant $-\lambda^2 \in \mathbb{R}$. We could've made the eigenvalue zero or positive, but it can be shown that these choices lead to solutions that can't satisfy the boundary conditions. Thus our constant must be real and negative. Furthermore, in order to satisfy the boundary conditions $X(0) = X(\pi) = 0$ we must have $= n \in \mathbb{N}$. Thus we obtain a family of solutions

$$X_n = \sin(nx), \;\; T_n = a_n \cos(nt) + b_n \sin(nt)$$

and by the principle of superposition, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \sin(nx).$$

Applying our initial condition we get

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(nx).$$

Since L is a self-adjoint operator with eigenfunctions $\sin(nx)$, we know that the eigenfunctions are orthogonal with respect to the L2 inner product. Therefore

$$\boxed{\int_0^\pi \sin(nx)\sin(mx)dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases}}$$

and so we get

$$a_n = \frac{2}{\pi}\int_0^\pi u_0(x)\sin(nx)dx.$$

Goal: show that the $\sin(nx)$ form a basis of eigenfunctions so we can conclude that we have found all possible solutions.

To begin, let us return to linear algebra and show that **symmetric matrices have a basis of eigenvectors.** We show this by showing that symmetric matrices have no generalized eigenvectors. Every square matrix has n linearly independent eigenvectors and generalized eigenvectors. Therefore if it has no generalized eigenvectors then its eigenvectors form a basis for $\mathbb{R}^n$.

Claim:

$$\boxed{(A-\lambda I)^2 X = 0 \implies (A-\lambda I)X = 0}$$

Proof:

$$0 = \langle X, (A-\lambda I)^2 X\rangle = \langle (A-\lambda I)X, (A-\lambda I)X\rangle = \|(A-\lambda I)X\|^2$$
$$\implies (A-\lambda I)X = 0.$$

We can do this inductively for general m:

$$(A-\lambda I)^m X = 0 \implies (A-\lambda I)^2(A-\lambda I)^{m-2}X = 0 \implies (A-\lambda I)^{m-1} = 0$$

and we can keep going to lower the exponent.

At this point, he went through a long-ish proof that for a symmetric compact operator L, it has eigenfunctions f_n that form a basis for the range of L. He then remarks that this seems useless to us because the operator we are interested in $L = \partial_x^2$ is not bounded. We don't prove this, but he claims that **if L is not compact, L^{-1} is compact.**

Green's functions.

If our ODE is

$$Lf = g$$

then we want to find the inverse of L so that we can solve it:

$$f = L^{-1}g.$$

Recall that we can break up the RHS into a bunch of "hits":

$$g = \int_0^1 g(y)\delta(x-y)dy$$

and then if we can find a solution $f(x;y)$ that solves

$$Lf = \delta(x-y)$$

we can obtain the solution:

$$f(x) = \int_0^1 g(y)f(x;y)dy.$$

We call such a function $f(x;y) =: G(x,y)$ a **Green's function**.

Example.

$$\begin{cases} f''(x) = g(x) \\ f(0) = f(1) = 0 \end{cases}$$

We look for a Green's function $G(x, y)$ that solves

$$G_{xx}(x, y) = \delta(x - y)$$

with boundary conditions $G(0, y) = G(1, y) = 0$. When $x < y$ and $x > y$, $\delta(x - y) = 0$ therefore our ODE looks like $G_{xx} = 0$ so we have

$$G(x, y) = \begin{cases} \alpha x, & x < y \\ \beta(x - 1), & x > y \end{cases}$$

after applying the boundary conditions. There are two conditions we enforce on G as $x \to y$. They are:

$$\boxed{\begin{cases} G(y^-, y) = G(y^+, y) \\ G_x(y^+, y) - G_x(y^-, y) = 1 \end{cases}}$$

The second condition comes from integrating our ODE once:

$$\int_{y^-}^{y^+} G_{xx}(x, y)dx = \int_{y^-}^{y^+} \delta(x - y)dx$$

$$\implies G_x(y^+, y) - G_x(y^-, y) = 1.$$

The first condition comes from the fact that, if G was discontinuous, then its first derivative G_x would be a δ, but this is not what we want - we want the second derivative to be a δ. So we need G to be continuous but have a slope discontinuity (a sharp corner). Enforcing these conditions, gives

$$\alpha y = \beta(y - 1), \quad \beta - \alpha = 1$$

$$\implies G(x, y) = \begin{cases} x(y - 1), & x \leq y \\ y(x - 1), & x \geq y \end{cases}$$

Maxwell's reciprocity principle.

If G is a Green's function of a symmetric linear operator, then

$$G(x, y) = G(y, x).$$

Proof:

$\langle G(x, y), LG(x, z)\rangle = \langle G(x, y), \delta(x-z)\rangle = G(z, y) = \langle LG(x, y), G(x, z)\rangle = \langle \delta(x-y), G(x, z)\rangle = G(y, z).$

Dynamical systems.

Consider autonomous ODEs of the form

$$\dot{x} = f(x, r).$$

If the ODE was not autonomous, we could just consider t as part of x, i.e. $x = (x_1, \cdots, x_n, t)$. A **fixed point** is a point x^* such that

$$\boxed{f(x^*) = 0.}$$

An important property of solutions to these ODEs is that they *cannot intersect themselves.* Because if they did, then at the point of intersection, f would have the same value as it did before so it would have to go in the same direction. In 1D this means that the solution on a line can never turn around - it can only go in one direction.

A fixed point $\bar{x}$ is called Lyapunov stable if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that if you start in $B_\delta(\bar{x})$ you stay in $B_\epsilon(\bar{x})$.

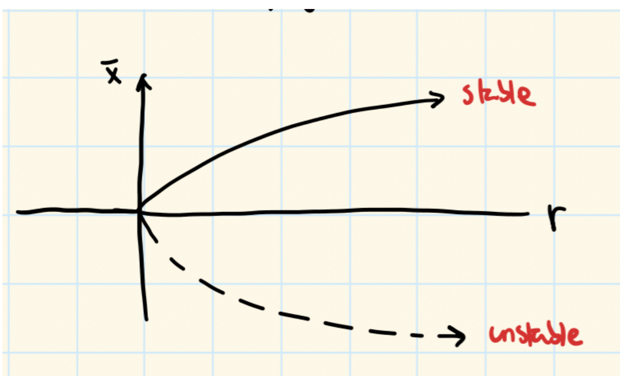
A stronger type of stability is when a fixed point is asymptotically stable, meaning that trajectories converge to this fixed point over time.

Saddle node bifurcation.

The canonical example of a saddle node bifurcation comes from the example

$$\boxed{\dot{x} = r - x^2.}$$

The fixed points are given by $x = \pm\sqrt{r}$ and we can easily see that the positive root is a stable fixed point and the negative root is unstable. Therefore the bifurcation diagram looks like



Example. Let θ and φ be the phases of two different fireflies. Then we have the following ODE governing their behavior:

$$\begin{cases} \dot{\theta} = \omega + \gamma \sin(\varphi - \theta) \\ \dot{\varphi} = \Omega. \end{cases}$$

From this we obtain

$$\dot{(\theta - \varphi)} = \omega - \Omega + \gamma \sin(\varphi - \theta)$$

and setting $\alpha = \theta - \varphi$ we have an autonomous ODE for α:

$$\dot{\alpha} = \delta - \gamma \sin \alpha$$

We can rescale time as $t \to -\gamma t$ to obtain

$$\dot{\alpha} = -\frac{\delta}{\gamma} - \sin \alpha$$

and then let $-\delta/\gamma = 1 + r$ to get

$$\dot{\alpha} = 1 + r - \sin \alpha.$$

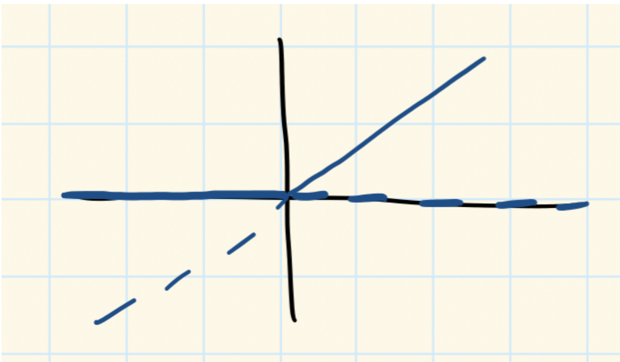
If $r > 0$, there are no fixed points. If $-2 < r < 0$, there are two fixed points. If $r = 0$, there is one fixed point. This is another example of a saddle node bifurcation.

Transcritical bifurcation.

The canonical example of a transcritical bifurcation is the ODE

$$\boxed{\dot{x} = x(r - x).}$$

This has fixed points $x = 0$ and $x = r$. The fixed points switch stability when they collide at $r = 0$:

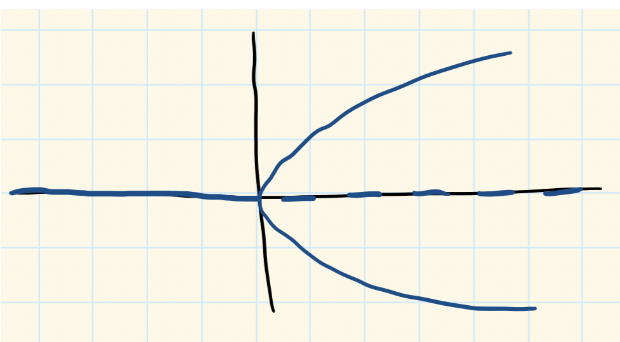


Pitchfork bifurcation.

The canonical example here is

$$\boxed{\dot{x} = x(\gamma - x^2).}$$

The fixed points are $x = 0$, $x = \sqrt{\gamma}$, and $x = -\sqrt{\gamma}$. When $\gamma \leq 0$ there is only one fixed point at $x = 0$. Then when $\gamma > 0$ we get two new fixed points that branch off, and the $x = 0$ fixed point switches stability.



2D flows.

We will consider 2D flows of the form

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y). \end{cases}$$

Near a fixed point, we can **linearize** the system:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

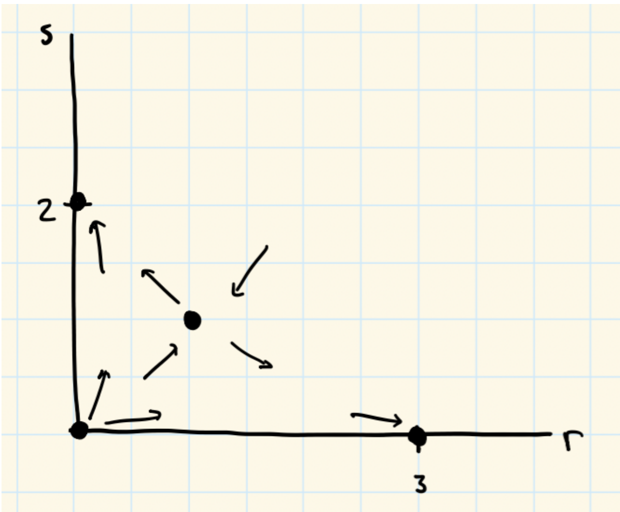
and look at eigenvalues.

Lotka-Volterra models.

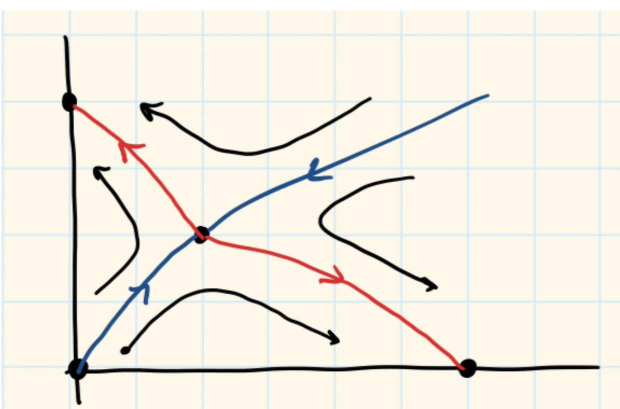
Example. Let r and s represent the number of rabbits and sheep, respectively. Then consider the system

$$\begin{cases} \dot{r} = r(3 - r - 2s) \\ \dot{s} = s(2 - s - r) \end{cases}$$

which has fixed points at $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(1, 1)$. Just by working through what direction the flow is moving near the fixed points, we get a sense for the stability of the points:



A saddle point is a fixed point with a **stable manifold** and an **unstable manifold.** A stable manifold can be found by starting near the point and reversing time to see what trajectory will be taken. An unstable manifold can be found by starting near the point and seeing what trajectory will be taken in forward time. Saddle points are unstable. Unless you start on a stable manifold, you may move towards it for some time but you will eventually move away.



So in this case we see that all trajectories end up with one of the species extinct!

Example. Let r and f represent the number of rabbits and foxes, respectively. Then consider

$$\begin{cases} \dot{r} = r(1-f) \\ \dot{f} = -f(1-r). \end{cases}$$

This system has fixed points $(0,0)$ and $(1,1)$. If $(r=0, f>0)$ then the trajectory points directly downward. If $(r>0, f=0)$, then the trajectory points directly toward the right. So we see that $(0,0)$ is a saddle point. To look at the behavior near $(1,1)$, we can rewrite our ODE in terms of $x := r-1$ and $y := f-1$. We obtain

$$(r \dot{-} 1) = (r-1+1)(1-f)$$

$$(f \dot{-} 1) = -(f-1+1)(1-r)$$

which is

$$\begin{cases} \dot{x} = -y(x+1) \\ \dot{y} = x(y+1) \end{cases}$$

We can linearize around $(x,y) = (0,0)$ to get

$$\begin{cases} \dot{x} \approx -y \\ \dot{y} \approx x \end{cases}$$

which gives oscillatory trajectories. Recall that *centers are not stable under nonlinear perturbations*. We can't know if this is the true behavior. Let us look for a conserved quantity. We return to our system for r and f and rewrite it as

$$\begin{cases} \frac{\dot{r}}{r} = 1-f \\ \frac{\dot{f}}{f} = r-1 \end{cases}$$

We multiply the top equation by $(r-1)$ and the bottom equation by $(1-f)$ and subtract to get

$$(r-1)\frac{\dot{r}}{r} - (1-f)\frac{\dot{f}}{f} = 0$$

$$\implies \dot{r} - \frac{\dot{r}}{r} - \frac{\dot{f}}{f} + \dot{f} = 0$$

$$\implies \frac{d}{dt}(r - \log r + f - \log f) = 0$$

so we have found a conserved quantity $F = r - \log r + f - \log f$. Thus the contour lines of F are the trajectories of the system. So indeed we do have a center with oscillatory trajectories.

Claim.

A system with a conserved quantity cannot have an asymptotically stable fixed point.

Explanation: conserved quantities are by definition functions $F(x,y)$ that do not depend on time $dF/dt = 0$ and that are not constant in any open set of the (x,y) domain (b/c if they were constant in an open set than it's trivial to say that they are conserved i.e. constant). If a fixed point is asymptotically stable then there would be some open set around it where all trajectories are converging to the fixed point and therefore the conserved quantity would have to be constant here (it would have to be equal to its value at the FP), which is a contradiction.

Index theory.

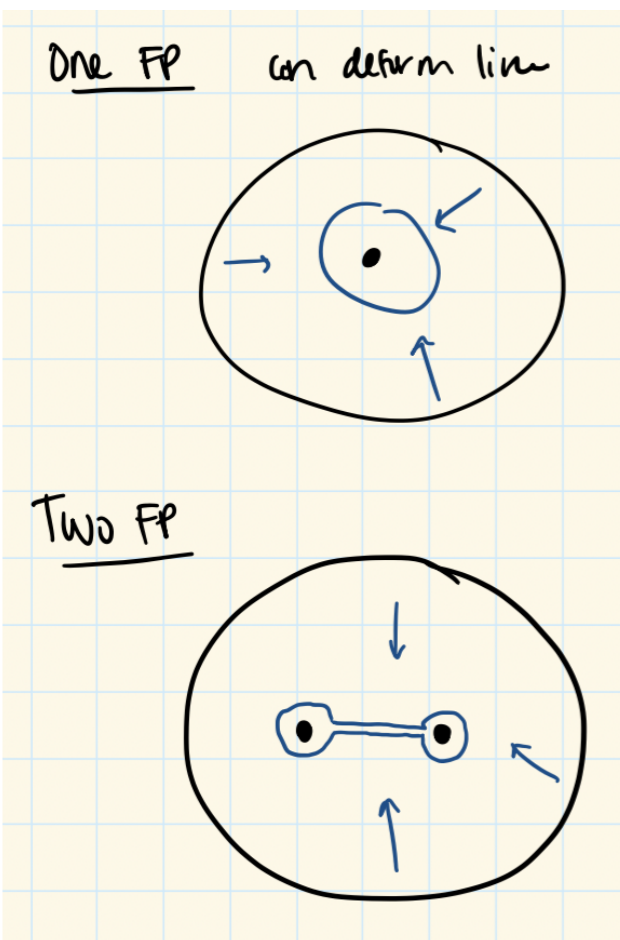
Again we consider a 2D system of autonomous ODEs

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

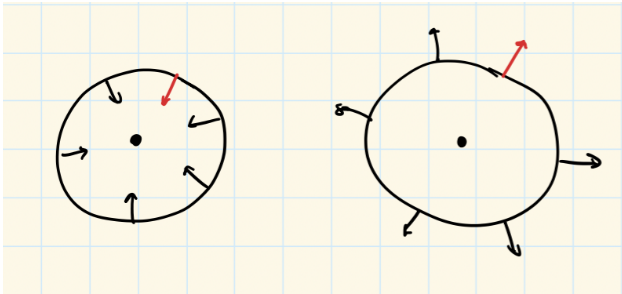
which can be written more succinctly as $\dot{\mathbf{x}} = v(\mathbf{x})$. Consider an arbitrary curve C in the (x,y) plane that does not go through any fixed points. We can choose a starting point along the curve and look at what direction our velocity vector $\dot{\mathbf{x}}$ is pointing here. We can look at **its angle from the horizontal.** Then we can move counter-clockwise around the curve and observe what direction this vector points as we go. Of course, when we return to our starting point, the vector will be pointing in the same direction, but the angle could have changed mod 2π. We call the number of rotations done by the velocity vector the **index of C**, and denote it

$$I(C) = \frac{[\varphi]}{2\pi}$$

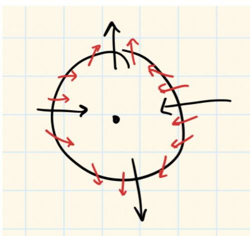
where $[\varphi]$ is the change in the velocity vector angle φ after one revolution. If we deform C in some continuous way (that doesn't cross any fixed points in the process), the index cannot change because its an integer and it can't just jump discontinuously to the next integer.



So to compute $I(C)$ we can simply look at the index number of small curves around the FPs. If the FP is unstable or stable, $I(C) = 1$. The velocity moves around one revolution counter-clockwise.



If we have a saddle point, $I(C) = -1$, i.e. the velocity vector moves around one revolution clockwise.



Poincaré-Bendixson theorem.

A limit cycle is a periodic orbit to which you converge or diverge from.

Example.

$$\begin{cases} \dot{r} = r(1-r) \\ \dot{\theta} = 1 \end{cases}$$

This has a fixed point at $r = 1$ that is stable so we get a limit cycle here.

Example. The Van der Pol oscillator is an ODE of the form

$$\ddot{x} + \nu(x^2 - 1)\dot{x} + x = 0.$$

We can turn this into a 2D ODE:

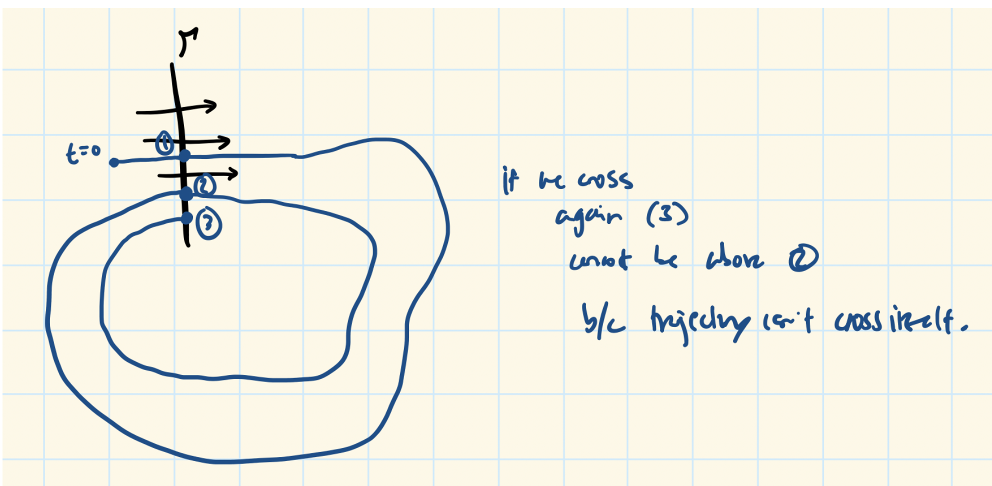
$$\begin{cases} \dot{x} = p \\ \dot{p} = -\nu(x^2 - 1)p - x \end{cases}$$

This has only one fixed point at $(x, p) = (0, 0)$. This FP is unstable so trajectories nearby are moving away from it. But when x and p are large, things are spiraling inwards. So we think there must be some limiting cycle between these regions.

Poincaré-Bendixson Theorem.

(Strogatz textbook) Suppose that:
(1) R is a closed, bounded subset of the plane,
(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuosly differentiable vector field on an open set containing R,
(3) R does not contain any fixed points, and
(4) there exists a trajectory C that is confined in R (i.e. it starts in R and stays there for all time).
Then either C is a closed orbit or it spirals towards a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

Proof: Esteban did not prove this rigorously but gave a handwavy argument. Consider a little piece of flow. We can draw a line that is transversal (not tangent) to the flow. Consider a trajectory that starts on one side of the line and crosses it. If it crosses again, it cannot cross at any point above the first crossing point because it cannot intersect itself.



Suppose we have a limiting point (not a FP) (I think this is just like a point that we get arbitrarily close to as $t \to \infty$ like it could be a point along a limit cycle). Near this point, the flow is non-zero so we can draw a line transverse to the flow that goes through the point. Then a trajectory that crosses the line Γ will have to come back to cross it again since it needs to get close to the limit point again. But by the above argument, it must come back to exactly the same place because otherwise it would get further and further from the point.

Example. Let us return to the Van der Pol oscillator,

$$\ddot{x} + \nu(x^2 - 1)\dot{x} + x = 0,$$

which has a FP at $(x, \dot{x}) = (0, 0)$ which is a source (unstable). We want to construct a trapping region such that if you start in it, you stay in it for all time. Then we can use Poincaré-Bendixson to conclude that there must be a limit cycle. Rewrite the ODE as

$$\frac{d}{dt}\left(\dot{x} + \nu\left(\frac{x^3}{3} - x\right)\right) + x = 0$$

and call $F(x) := x^3/3 - x$. Let

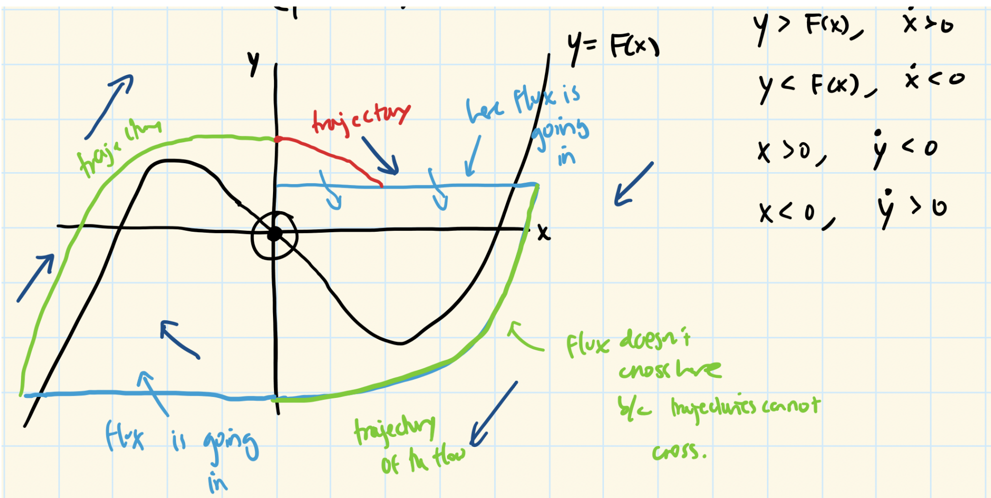
$$w := \dot{x} + \nu F(x)$$

then our system is

$$\begin{cases} \dot{w} = -x \\ \dot{x} = w - \nu F(x). \end{cases}$$

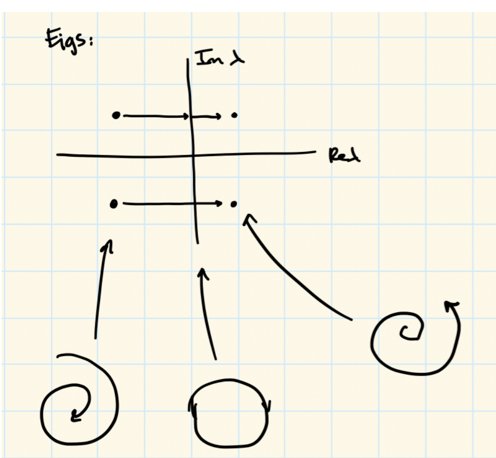
Let $y = w/\nu$, then the system becomes

$$\begin{cases} \dot{y} = -x/\nu \\ \dot{x} = \nu(y - F(x)) \end{cases}$$



Hopf bifurcation.

If the imaginary part of the eigenvalues switch in sign then we get a bifurcation in which the stability of a spiral switches.



Example.

$$\begin{cases} \dot{r} = -\nu r - r^3 = -r(\nu + r^2) \\ \dot{\theta} = 1 \end{cases}$$

If $\nu > 0$, then there is one fixed point at $r = 0$ and it is attractive. If $\nu < 0$ then the FP at $r = 0$ is repulsive and the FP at $r = \sqrt{-\nu}$ is attractive. Here we have a limit cycle appearing out of nowhere.

Example.

$$\begin{cases} \dot{r} = -\nu r + r^3 - r^5 \\ \dot{\theta} = 1 \end{cases}$$

More dimensions.

Chaos describes the phenomena in which small perturbations in the initial data lead to completely different solutions.

Lorenz 63 system.

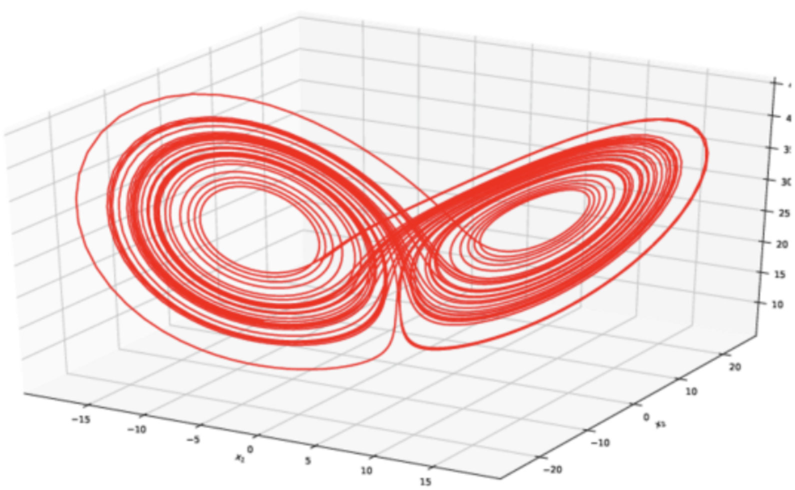
Consider the following system of ODEs, from 1963,

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = (r - z)x - y \\ \dot{z} = xy - bz, \end{cases}$$

where $\sigma, r, b > 0$ are positive parameters. Clearly, $(0,0,0)$ is a fixed point. There are also two more fixed points given by

$$\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right).$$

If we solve the system numerically, we will see the trajectory converging to a strange attractor: a manifold to which solutions converge, but the behavior is weird - not periodic or normal. The trajectories may look like they are crossing, but they're not because this is in 3D.



If we let $\mathbf{x} = (x, y, z)$ and denote $\dot{\mathbf{x}} = \mathbf{v}$, we can compute

$$\nabla \cdot \mathbf{v} = -\sigma - 1 - b < 0$$

and therefore by the divergence theorem,

$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

where V is the volume. So if the volume is going to zero, then our statement about trajectories not crossing because we are in 3D doesn't make as much sense.

If we take two initial conditions that are close together,

$$\|x_1(0) - x_2(0)\| < \epsilon$$

then after time t, their distance will have grown exponentially:

$$\|x_1(t) - x_2(t)\| \sim \epsilon e^{Lt}$$

where L is the Lyapunov exponent.

Hamiltonian systems.

Recall that Newton's law tells us that force is equal to mass times acceleration. If the force is the gradient of some potential, this gives

$$m_i \ddot{x}_i = F_i = -\nabla_{x_i} V(x).$$

We know that energy,

$$E = \sum_i \frac{1}{2} m_i \|\dot{x}_i\|^2 + V(x),$$

is conserved. Denote the momentum as

$$p_i = m_i \dot{x}_i$$

then we can write energy as

$$H(p, x) = \sum_i \frac{\|p_i\|^2}{2m_i} + V(x)$$

and then we obtain the following system of equations for x_i and p_i:

$$\boxed{\begin{cases} \dot{p}_i = -\nabla_{x_i} H \\ \dot{x}_i = \nabla_{p_i} H \end{cases}}$$

Observe that

$$\dot{H} = \sum_i \left(\nabla_{x_i} H \cdot \dot{x}_i + \nabla_{p_i} H \cdot \dot{p}_i \right) = 0$$

so the **Hamiltonian is conserved.**

Poincaré's Recurrence Theorem.

For a Hamiltonian system bounded in some finite domain, for any open set in the domain, a flow that passes through this open set at some point will pass through it infinitely often. In other words, you will get arbitrarily close to any place you have been in the past, infinitely often.

As in methods of applied math, we can look at plane wave solutions to systems of the form,

$$e^{i(kx - \omega t)},$$

and plug these into our system to find a *dispersion relation.* More generally, we can consider solutions of the form

$$A(x,t)e^{i\theta(x,t)}$$

and define the wave number and phase via the formulas

$$\begin{cases} k = \nabla_x \theta \\ \omega = -\theta_t \end{cases}$$

which is consistent with the plane wave ansatz. We expect that locally ω still satisfies a dispersion relation, but now it may depend on space and time

$$\omega = \Omega(k, x, t).$$

Taking cross derivatives we obtain

$$0 = k_t + \nabla_x \omega = k_t + \nabla_k \Omega \cdot \nabla_x k + \nabla_x \Omega$$

$$\implies k_t + \nabla_k \Omega \cdot \nabla_x k = -\nabla_x \Omega$$

If we consider the wave number along a path $x(t)$: $k(x(t), t)$, we have

$$\frac{dk}{dt} = \nabla_x k \cdot \dot{x} + k_t$$

now we see that if we let $\dot{x} = \nabla_k \Omega$ we obtain what looks like a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla_k \Omega \\ \dot{k} = -\nabla_x \Omega \end{cases}$$

where now our Hamiltonian is the dispersion relation.

Returning to our original Hamiltonian system,

$$\begin{cases} \dot{p_i} = -\nabla_{x_i} H \\ \dot{x_i} = \nabla_{p_i} H, \end{cases}$$

we can now ask ourselves if there is some $s(x,t)$ (above it was our θ), such that

$$\begin{cases} p = \nabla_x s \\ H = -s_t \end{cases}$$

Calculus of variations.

Consider a function $L(\dot{x}, x)$ called the Lagrangian. We want to compute its integral along a path $x(t)$ from t_0 to t_1 and find the path that minimizes this value:

$$\min_{x(t)} \int_{t_0}^{t_1} L(\dot{x}, x) dt.$$

Consider perturbing the path slightly

$$x(t) + \eta(t)$$

where $\eta(t_0) = \eta(t_1) = 0$. Then our integral becomes

$$\begin{aligned}\int_{t_0}^{t_1} L(\dot{x}+\dot{\eta}, x+\eta)dt &= \int_{t_0}^{t_1} L(\dot{x},x) + \frac{\partial L}{\partial \dot{x}}(\dot{x},x)\dot{\eta} + \frac{\partial L}{\partial x}(\dot{x},x)\eta + \cdots dt \\ &= \int_{t_0}^{t_1} L(\dot{x},x)dt + \int_{t_0}^{t_1} \left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x}\right)\eta dt + \left[\frac{\partial L}{\partial \dot{x}}\eta\right]_{t_0}^{t_1}\end{aligned}$$

The last term is zero because of the boundary conditions on η. If $x(t)$ is the minimizer of our functional, then any perturbations around it should only serve to increase the functional. However, since η can be anything, subject to boundary conditions, if the second integrand is not zero, it would be possible to choose η s.t. the second term is negative. From this idea, we obtain the...

Euler-Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

Now, back to our problem of finding an $s(x,t)$ as described above. If we set

$$s(x,t) = \min_{x(t)} \int_{t_0}^{t_1} L(\dot{x},x)dt$$

where $x(t_0) = x$ and $x(t) = x$ (i.e. x and t tell you up to where and when you are integrating), then we see that this is just what we were looking for.

$\vdots$

Solving some ODEs.

Integrating factor.

If we have a first order ODE with variable coefficients of the form

$$\boxed{\dot{x}(t) + a(t)x(t) = b(t)},$$

then we can multiply the whole thing by an integrating factor $m(t)$ and choose $m(t)$ such that

$$\frac{d}{dt}(m(t)x(t)) = \dot{m}x + m\dot{x} = amx + m\dot{x}$$

$$\implies \dot{m}(t) = a(t)m(t)$$

$$\implies \boxed{m(t) = e^{\int_0^t a(s)ds}}$$

Then the ODE has become

$$\frac{d}{dt}(e^{\int_0^t a(s)ds}x(t)) = e^{\int_0^t a(s)ds}b(t)$$

and so we have solution

$$x(t) = e^{-\int_0^t a(s)ds}\int_0^t b(\tau)e^{\int_0^\tau a(s)ds}d\tau.$$

Bernoulli.

For an ODE of the form

$$\boxed{\dot{x} = a(t)x + b(t)x^p}$$

we make the substitution

$$\boxed{y = x^{1-p}}$$

then this transforms the ODE into

$$\begin{aligned}\dot{y} &= (1-p)x^{-p}\dot{x} \\ &= (1-p)x^{-p}(a(t)x + b(t)x^p) \\ &= (1-p)(a(t)y + b(t))\end{aligned}$$

which is a first order variable coefficient ODE that can now be solved with integrating factors.

Ricatti.

For a nonlinear ODE of the form

$$\boxed{\dot{x} = a(t)x^2 + b(t)x + c(t)}$$

we can make a guess that the solution looks like

$$\boxed{x = -\frac{\dot{y}}{ay}}$$

and then the ODE for y is given by

$$\ddot{y} - (b + \frac{\dot{a}}{a})\dot{y} + acy = 0$$

which is now second order, but it is **linear!**

Autonomous second order ODE.

Consider, for example, the ODE

$$y''(x) + y^2 = 1.$$

Trick: give up autonomy to reduce the order by 1. Let

$$\boxed{y'(x) = u(y)}$$

then

$$y''(x) = u'(y)y'(x) = u'(y)u$$

and so the equation becomes

$$uu' + y^2 = 1.$$