PDE 1 Notes

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Professor: Scott Armstrong, Book: PDE (2nd Edition, Evans)

Chapter 2: Four Important Linear Partial Differential Equations

2.1. Transport Equation.

The transport equation is

$$u_t + b \cdot Du = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \tag{1}$$

where $b = (b_1, \cdots, b_n)$ and $u(x,t)$ is the unknown. Recall that a **directional derivative** of a function f measures the rate at which f changes in a particular direction v, and is given by $v \cdot \nabla f$. The transport equation tells us that the directional derivative of $u(x,t)$ in the direction $(b, 1)$ is zero, i.e. u does not change in this direction. Therefore if we start at any point (x,t) and move along the line $(x + sb, t + s)$, $s \in \mathbb{R}$, then u remains constant:

$$\boxed{u(x,t) = u(x + sb, t + s), \quad s \in \mathbb{R}.}$$

2.1.1. Initial-value problem.

If we have initial data $u(x,0) = g(x)$ then we must trace the line $(x + sb, t + s)$ back to when $t = 0$, which is when $s = -t$. Here, $x + sb = x - bt$ so we have

$$u(x,t) = u(x - bt, 0) = g(x - bt). \tag{2}$$

Existence and uniqueness of solutions to the IVP are obvious: if g is C^1 then indeed (2) is a solution, and if the IVP has a sufficiently regular solution, it must certainly be given by (2). Even if g is not C^1, we still define (2) to be a weak solution (more on that later).

2.1.2. Nonhomogeneous problem.

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{3}$$

Fix $(x,t) \in \mathbb{R}^n$ and set

$$z(s) = u(x + bs, t + s), \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned}
u(x,t) - g(x-tb) = z(0) - z(-t) &= \int_{-t}^{0} \dot{z}(s)ds \\
&= \int_{-t}^{0} (b \cdot Du + u_t)(x+sb, t+s)ds \\
&= \int_{-t}^{0} f(x+sb, t+s)ds \\
&= \int_{0}^{t} f(x+(s-t)b, s)ds
\end{aligned}$$

and so we get the solution

$$\boxed{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)ds.} \tag{4}$$

This formula is an instance of Duhamel's principle - the first term is the homogeneous solution with the actual initial data. The integrand of the second term

$$a(x,t;s) = f(x+(s-t)b, s)$$

solves

$$\begin{cases} a_t + b \cdot Da = 0 \\ a(x,s) = f(x,s). \end{cases}$$

2.2. Laplace's Equation.

We have Laplace's equation

$$\triangle u = 0 \tag{5}$$

and Poisson's equation

$$-\triangle u = f. \tag{6}$$

In both we assume $x \in U$ where $U \subset \mathbb{R}^n$ is a given open set.

Harmonic function.

A C^2 function satisfying Laplace's equation.

Physical interpretation. If u denotes the density of some quantity in *equilibrium* and V is a smooth subregion in U then the net flux through the boundary is zero:

$$\boxed{\int_{\partial V} F \cdot \nu dS = 0}$$

where F denotes the flux density.

Divergence Theorem.

If $u \in C^1(\bar{U})$,

$$\int_U D \cdot u dx = \int_{\partial U} u \cdot \nu dS$$

From this we conclude that

$$\int_V (D \cdot F) dx = 0$$

and therefore

$$D \cdot F = 0$$

in U since V was arbitrary. In many cases, the flux density is proportional to the gradient Du, but pointing in the opposite direction, so we get Laplace's equation,

$$-\triangle u = 0.$$

2.2.1. Fundamental Solution.

Fundamental solution of Laplace's equation.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{7}$$

defined for $x \neq 0$. Recall $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

We can motivate these formulas by noting that Laplace's equation is invariant under rotations, i.e. if $\triangle u = 0$, O is an orthogonal $n \times n$ matrix, and we define $v(x) = u(Ox)$, then $\triangle v = 0$. Therefore we seek a **radially symmetric** solution having the form

$$u(x) = v(r).$$

We get that

$$\boxed{\triangle u = v''(r) + \frac{n-1}{r} v'(r)}$$

Laplacian of radial function.

so Laplace's equation is satisfied if and only if

$$v'' + \frac{n-1}{r} v' = 0.$$

If $v' \neq 0$ we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r}.$$

Integrating both sides gives

$$\log(|v'|) = (1-n)\log|r| + c \iff |v'| = a|r|^{1-n} \iff v' = ar^{1-n}$$

so

$$v(r) = \begin{cases} b\log r + c & (n = 2) \\ br^{2-n} + c & (n \geq 3) \end{cases}$$

where b, c are constants.

We have the following estimates:

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \ \ |D^2\Phi(x)| \leq \frac{C}{|x|^n} \tag{8}$$

Solution to Poisson's equation.

Suppose $f \in C_c^2(\mathbb{R}^n)$. Then

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$$

satisfies
(i) $u \in C^2(\mathbb{R}^n)$
(ii) $-\triangle u = f$ in $\mathbb{R}^n$.

The fundamental solution can be interpreted as "solving"

$$-\triangle \Phi = \delta_0.$$

2.2.2. Mean-value formulas.

Mean-value formulas for Laplace's equation.

If $u \in C^2(U)$ is harmonic, then

$$u(x) = \fint_{\partial B(x,r)} udS = \fint_{B(x,r)} udy.$$

Proof of first equality:

1. Set
$$\phi(r) := \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{\partial B(0,1)} u(x+rz)dS(z).$$

2. Then
$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} Du(x+rz)\cdot zdS(z) \\
&= \fint_{\partial B(x,r)} Du(y)\cdot \frac{y-x}{r}dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}dS(y) \\
&= \frac{r}{n}\fint_{B(x,r)} \triangle u(y)dy \qquad \text{(Green's formula)} \\
&= 0.
\end{aligned}$$

3. Since ϕ is constant,
$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} u(y)dS(y) = u(x).$$

In short,

1. Take the average value of u over surface of ball and change coordinates to unit ball.

2. Differentiate with respect to radius, change coordinates back to $B(x,r)$, and use Green's formula.

3. Since this is constant with respect to radius, let $r \to 0$.

The second equality follows from the first using **polar coordinates**:

$$\begin{aligned}\int_{B(x,r)} u(y)dy &= \int_0^r \int_{\partial B(x,s)} u(y)dS(y)ds \\ &= \int_0^r u(x)n\alpha(n)s^{n-1}ds \\ &= \alpha(n)r^n u(x).\end{aligned}$$

<u>Aside:</u> you could prove the mean-value property in the 2D case by arguing as follows. Let $u(x,y)$ be harmonic. Then it must be the real part of some analytic complex function $f(z) = u(x,y) + iv(x,y)$. A direct consequence of Cauchy integral formula is the mean-value property,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + Re^{i\theta})d\theta.$$

And separating the real and imaginary parts gives

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + R\cos\theta, y + R\sin\theta)d\theta.$$

Converse to mean-value property.

If $u \in C^2(U)$ satisfies

$$u(x) = \fint_{\partial B(x,r)} udS$$

for each ball $B(x,r) \subset U$ then u is harmonic.

Proof by contradiction: If $\triangle u$ is not identically equal to zero then there exists some ball $B(x,r) \subset U$ such that, say, $\triangle u > 0$ within $B(x,r)$. But then for ϕ in the above proof,

$$0 = \phi'(r) = \frac{r}{n}\fint_{B(x,r)} \triangle u(y)dy > 0,$$

a contradiction.

Green's formulas.

Let $u, v \in C^2(\bar{U})$. Then
(i) $\int_U \triangle u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$
(ii) $\cdots$
(iii) $\cdots$

(i) is just divergence theorem applied to Du.

2.2.3. Properties of harmonic functions.

All of the following observations are deduced from the mean-value formulas.

Strong maximum principle.

Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic in U. Then

(i) (weak) $\max_{\bar{U}} u = \max_{\partial U} u$
(ii) (strong) If $\exists x_0 \in U$ s.t. $u(x_0) = \max_{\bar{U}} u$ then u is constant in U.

Note: replacing u by $-u$ we recover equivalent statements about the minimum.

Proof by contradiction: suppose $\exists x_0 \in U$ s.t. $u(x_0) = M := \max_{\bar{U}} u$. Then by the mean-value property,

$$M = u(x_0) = \fint_{\partial B(x_0,r)} u(y) dS(y)$$

for all $r > 0$. Let r be small enough such that $B(x_0, r) \subset U$. Then if there is any point $y \in \partial B(x_0, r)$ for which $u(y) < M$, then the whole RHS would be $< M$ and so we get $M < M$, therefore it must be that $u(y) = M$ for all $y \in \partial B(x_0, r)$. We continue in this way moving around U to conclude that u must be constant. This proves (ii), which implies (i).

Positivity.

If

$$\begin{cases} \triangle u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where $g \geq 0$, then u is positive everywhere in U if g is positive somewhere on ∂U.

Let $m = \min_{\bar{U}} g$ and $M = \max_{\bar{U}} g$. If $m = M$ then g is constant and so u is equal to the same positive constant. Otherwise, since u cannot attain its minimum inside U, $u > m \geq 0$ in U.

Uniqueness.

Let $g \in C(\partial U), f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary-value problem

$$\begin{cases} -\triangle u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Suppose u_1 and u_2 solve the BVP. Let $\omega_\pm := \pm(u_1 - u_2)$ and apply weak maximum principle to get

$$u_1 - u_2 \leq 0 \text{ and } -(u_1 - u_2) \leq 0 \implies u_1 - u_2 \geq 0$$

and conclude that $u_1 - u_2 \equiv 0$ in U.

Smoothness.

If $u \in C(U)$ satisfies the mean-value property for each ball $B(x, r) \subset U$, then

$$u \in C^\infty(U).$$

Standard mollifier.

Define the standard mollifier $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x^2|-1}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{9}$$

with $C > 0$ chosen so that $\int_{\mathbb{R}^n} \eta = 1$. For each $\epsilon > 0$ set

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right). \tag{10}$$

These functions are $C^\infty(\mathbb{R}^n)$ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = 1, \;\; \operatorname{spt}(\eta_\epsilon) \subset B(0, \epsilon).$$

If $f : U \to \mathbb{R}$ is locally integrable, its mollification $f^\epsilon := f * \eta_\epsilon$ satisfies
(i) $f^\epsilon \in C^\infty(U_\epsilon)$
(ii) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$

Proof idea: Let $\epsilon > 0$ and let $u^\epsilon = u * \eta_\epsilon$. Show that $u^\epsilon \equiv u$ in U_ϵ and so $u \in C^\infty(U_\epsilon)$ for all $\epsilon > 0$.

Estimates on derivatives.

Assume u is harmonic in U. Then

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

for each ball $B(x_0, r) \subset U$ and each multiindex α of order $|\alpha| = k$. Here,

$$C_0 = \frac{1}{\alpha(n)}, \;\; C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}.$$

Proof outline: induction on k. The case for $k = 0$ is immediate from mean-value property.

Liouville's Theorem.

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof: From previous theorem,

$$\begin{aligned} |Du(x_0)| &\leq \frac{\sqrt{n}C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \\ &\leq \frac{\sqrt{n}C_1\alpha(n)r^n}{r^{n+1}} \|u\|_{L^\infty(B(x_0,r))} \\ &\leq \frac{\sqrt{n}C_1\alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0 \end{aligned}$$

as $r \to \infty$. So $Du \equiv 0$ and u is constant.

Representation formula.

Let $f \in C_c^2(\mathbb{R}^n), \mathbf{n \geq 3}$. Then any bounded solution of

$$-\triangle u = f \text{ in } \mathbb{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy + C$$

for some constant C.

Since $\Phi(x) \to 0$ as $|x| \to \infty$, $\tilde{u} = \Phi * f$ is a bounded solution **(why bdd?)** of $-\triangle u = f$ in $\mathbb{R}^n$. If u is another bounded solution, $w := \tilde{u} - u$ is harmonic and bounded and thus constant by Liouville's.

Analyticity.

Assume u is harmonic in U. Then u is analytic in U.

Need to show that u can be represented by a convergent power series in some neighborhood of x_0.

Harnack's inequality.

For each connected open set $V \subset\subset U$, there exists $C > 0$, depending only on V, such that

$$\sup_V u \leq C \inf_V u$$

for all non-negative harmonic functions u in U.

2.2.4. Green's function.

We derive a Green's function that allows us to write down a general representation formula for the solution to the **Poisson boundary value problem**,

$$\begin{cases} -\triangle u = f & u \in U \\ u = g & u \in \partial U. \end{cases} \tag{11}$$

For fixed x, define a corrector function $\phi^x(y)$ solving the BVP,

$$\boxed{\begin{cases} \triangle \phi^x = 0 & \text{in } U \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U. \end{cases}}.$$

Green's function.

$$G(x,y) := \Phi(y-x) - \phi^x(y), \qquad x, y \in U, x \neq y$$

Derivation:

Representation formula using Green's function.

If $u \in C^2(\bar{U})$ solves (11), then

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y) dS(y) + \int_U f(y) G(x,y) dy$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y) \cdot \nu(y).$$

- Now we have a formula for the solution of the BVP if we can construct the Green's function G. This is in general only possible when U has simple geometry, such as for the **half-space and the unit ball.**

- We can think of G as a function of y, solving

$$\begin{cases} -\triangle G = \delta_x & \text{in } U \\ G = 0 & \text{on } \partial U. \end{cases}$$

- The Green's function is symmetric in x and y, i.e. for all $x \neq y$,

$$G(x,y) = G(y,x).$$

2.2.5. Energy methods.

In contrast to using explicit representation formulas, here we use some **energy methods**, i.e. techniques incolving the L2 norms of various expressions.

An alternative proof of **uniqueness** of $C^2(\bar{U})$ solutions to

$$\begin{cases} -\triangle u = f \\ u = g \end{cases} \tag{12}$$

is as follows. Assume u and $\tilde{u}$ are both solutions. Set $w := u - \tilde{u}$. Then $\triangle w = 0$ in U and $w = 0$ on ∂U, so by integration by parts,

$$0 = -\int_U w \triangle w dx = \int_U |\nabla w|^2 dx$$

therefore since $w = 0$ on the boundary and does not change, $w \equiv 0$.

Dirichlet's Principle.

Define the **energy functional**

$$I[w] := \int_U \frac{1}{2}|\nabla w|^2 - wf dx, \tag{13}$$

for w belonging to the **admissible set**

$$\mathcal{A} := \{w \in C^2(\bar{U}) | w = g \text{ on } \partial U\}. \tag{14}$$

Then $u \in C^2(\bar{U})$ solves (12) if and only if

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Proof:
($\Rightarrow$) Suppose u solves (12). Choose $w \in \mathcal{A}$. Then

$$\begin{aligned} 0 &= \int_U (-\triangle u - f)(u - w) dx \\ &= \int_U \nabla u \cdot \nabla u - \nabla u \cdot \nabla w - f(u - w) dx \end{aligned}$$

and so

$$\begin{aligned} \int_U |\nabla u|^2 - uf \, dx &= \int_U \nabla u \cdot \nabla w - wf \, dx \\ &\leq \int_U |\nabla u||\nabla w| - wf \, dx \quad \text{(Cauchy-Schwarz)} \\ &\leq \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2 - wf \, dx \quad \text{(Cauchy inequality)} \end{aligned}$$

Subtracting $\int_U \frac{1}{2}|\nabla u|^2 dx$ from both sides completes this direction.

($\Leftarrow$) Suppose u minimizes the energy functional on the admissible set. Let $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v]$$

where $\tau \in \mathbb{R}$. Since $u + \tau v \in \mathcal{A}$ for all τ, the scalar function $i(\tau)$ has a minimum at $\tau = 0$ and so $i'(0) = 0$. We have

$$\begin{aligned} i(\tau) &= \int_U \frac{1}{2}|\nabla(u + \tau v)|^2 - (u + \tau v) f \, dx \\ &= \int_U \frac{1}{2}|\nabla u|^2 + \tau \nabla u \cdot \nabla v + \frac{1}{2}\tau^2 |\nabla v|^2 - (u + \tau v) f \, dx \end{aligned}$$

and so

$$\begin{aligned} 0 &= i'(0) \\ &= \int_U \nabla u \cdot \nabla v - vf \; dx \\ &= \int_U v(-\triangle u - f) dx \end{aligned}$$

and since this holds for all $v \in C_c^\infty(U)$ we have $-\triangle u = f$.

2.3. Heat Equation.

Now we study the heat equation

$$u_t - \triangle u = 0 \tag{15}$$

and the inhomogeneous heat equation

$$u_t - \triangle u = f. \tag{16}$$

Guiding principle: solutions to the heat equation will yield assertions that are similar (but more complicated) to those for harmonic functions.

Physical interpretation. The heat equation describes how the density of some quantity (heat, chemical concentration, etc.) evolves in time. If $V \subset U$ is a smooth subregion, then the rate of change of the total quantity within V is equal to the negative net flux through ∂V:

$$\boxed{\frac{d}{dt}\int_V u \; dx = -\int_{\partial V} F \cdot \nu dS}$$

where F is the flux density. Using the divergence formula, we obtain

$$u_t = -\nabla \cdot F.$$

In many situations,

$$F = -\alpha \nabla u$$

since the flow is from regions of higher to lower concentration. Substituting this in, we obtain the heat equation.

2.3.1. Fundamental solution.

Fundamental solution of the heat equation.

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$$

Derivation: we observe that since (15) involves one derivative in time and two in space, if $u(x,t)$ is a solution, then so is

$$u(\lambda x, \lambda^2 t).$$

This is called *parabolic* scaling of space-time. This motivates us to look for a *self-similar* solution that satisfies

$$\boxed{u(x,t) = \lambda^\alpha u(\sqrt{\lambda} x, \lambda t).}$$

We can eliminate the dependence on the second variable by setting

$$\lambda = \frac{1}{t},$$

which gives

$$u(x,t) = \frac{1}{t^\alpha} u\left(\frac{x}{\sqrt{t}}, 1\right) = \frac{1}{t^\alpha} v\left(\frac{x}{\sqrt{t}}\right).$$

Call $z := x/\sqrt{t}$. Plugging this into the heat equation gives

$$\begin{aligned} 0 &= u_t - \triangle u \\ &= -\alpha t^{-\alpha-1} v(z) + t^{-\alpha} \nabla v(z) \cdot \left(-\frac{1}{2} x t^{-3/2}\right) - t^{-\alpha} \triangle v t^{-1} \\ \implies 0 &= -\alpha v(z) - \frac{1}{2} z \cdot \nabla v(z) - \triangle v(z). \end{aligned}$$

We simplify further by guessing that v is *radially symmetric*. The second term becomes

$$\begin{aligned} z \cdot \nabla v(z) &= z \cdot \nabla_z v(|z|) \\ &= z \cdot \frac{v'(r) z}{r} \\ &= r v'(r) \end{aligned}$$

and the third term becomes

$$\begin{aligned} \triangle v(z) &= \sum_{i=1}^n v_{z_i z_i} \\ &= \sum_{i=1}^n \frac{\partial}{\partial z_i}\left(\frac{v'(r) z_i}{r}\right) \\ &= \frac{n-1}{r} v'(r) + v''(r) \end{aligned}$$

so our ODE for $v(r)$ is

$$0 = \alpha v + \frac{1}{2} r v'(r) + \frac{n-1}{r} v'(r) + v''(r).$$

Choose $\alpha = n/2$, multiply by r^{n-1}, and rewrite to get

$$\begin{aligned} (r^{n-1} v')' + \frac{1}{2}(r^n v)' &= 0 \\ \implies r^{n-1} v' + \frac{1}{2} r^n v &= C \quad \text{(choose } C = 0\text{)} \\ \implies v'(r) &= -\frac{1}{2} r v \\ \implies v(r) &= A e^{-\frac{|x|^2}{4t}} \\ \implies u(x,t) &= \frac{A}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \end{aligned}$$

We choose the constant A such that the integral of the fundamental solution is normalized. For each time $t > 0$,

$$\boxed{\int_{\mathbb{R}^n} \Phi(x,t)dx = 1.}$$

Proof:

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(x,t)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\
&= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} dz \\
&= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-z_i^2} dz_i \\
&= \frac{\sqrt{\pi}^n}{\pi^{n/2}} = 1.
\end{aligned}$$

Solution of initial-value problem

Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define u by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy.$$

Then u solves

$$\begin{cases} u_t - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

Proof: Clearly u solves the heat equation since the fundamental solution solves the heat equation,

$$u_t - \triangle u = \int_{\mathbb{R}^n} (\Phi_t - \triangle_x \Phi)(x-y,t)g(y)dy = 0.$$

The width of Φ scales like $\sqrt{t}$ and the height scales like $\frac{1}{t^{n/2}}$, so we can see that $\Phi(x,t) \to \delta(x)$ as $t \to 0$.

- We can think of the fundamental solution Φ as solving

$$\boxed{\begin{cases} \Phi_t - \triangle \Phi = 0 \\ \Phi(x,0) = \delta(x) \end{cases}}$$

- The heat equation forces **infinite propagation speed:** if g is bounded, continuous, $g \geq 0$ and $g > 0$ somewhere, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy > 0$$

for all $t > 0$. (because for any fixed value of $x \in \mathbb{R}^n$, the integrand is a Gaussian centered at x times g and the Gaussian is > 0 everywhere.)

Solution of nonhomogeneous problem.

Let

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy ds.$$

Then u solves

$$\begin{cases} u_t - \triangle u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Derivation: Fix $0 < s < t$, then define

$$u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy.$$

This solves the IVP

$$\begin{cases} u_t - \triangle u = 0 \\ u(x,s) = f(x,s). \end{cases}$$

Duhamel's Principle tells us that integrating $u(x,t;s)$ from $s=0$ to $s=t$ will then solve the heat equation with f on the RHS, i.e. our solution is

$$\int_0^t u(x,t;s) ds.$$

Solution of nonhomogeneous with initial data.

Combining the previous two theorems, we have that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy ds$$

solves

$$\begin{cases} u_t - \triangle u = f \\ u(x,0) = g(x). \end{cases}$$

2.3.2. Mean-value formula.

Definitions.

(i) We define the parabolic cylinder

$$U_T := U \times (0, T].$$

(ii) The parabolic boundary of U_T is

$$\Gamma_T := \bar{U}_T - U_T.$$

It comprises the bottom and sides of the cylinder.

(iii) For fixed $x \in \mathbb{R}^n, t \in \mathbb{R}, r > 0$ we define the "heat ball,"

$$E(x,t;r) := \left\{ (y,s) \in \mathbb{R}^{n+1} | s \leq t, \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}.$$

This is a region in space-time, the boundary of which is a level set of $\Phi(x-y, t-s)$.

For Laplace's equation, we computed the mean over spheres $\partial B(x,r)$, which were levels sets of the fundamental solution $\Phi(x-y)$ (since it is radially symmetric). Analogously, we now consider heat balls $E(x,t;r)$ which are levels sets of the fundamental solution $\Phi(x-y, t-s)$.

Mean-value property for the heat equation.

Let $u \in C_1^2(U_T)$ solve the heat equation. Then

$$u(x,t) = \frac{1}{4r^n} \int \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for each $E(x,t;r) \subset U_T$.

We did not cover the proof of this in class.

2.3.3. Properties of solutions.

We really did not cover any of this in class. We were told that analogous results to the Laplace equation exist and that they were more complicated.

2.3.4. Energy methods.

Uniqueness.

There exists only one solution $u \in C_1^2(\bar{U}_T)$ of the initial-boundary-value problem

$$\begin{cases} u_t - \triangle u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases}$$

Proof: If u_1 and u_2 are both solutions, then $w = u_1 - u_2$ solves

$$\begin{cases} w_t - \triangle w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Set

$$e(t) := \int_U w^2(x,t)dx,$$

then

$$\dot{e}(t) = \int_U 2w(x,t)w_t(x,t)dx = 2\int_U w\triangle w dx = -2\int_U |\nabla w|^2 dx \leq 0$$

so

$$e(t) \leq e(0) = 0 \implies e(t) = 0 \implies w \equiv 0.$$

Backwards uniqueness.

Suppose u_1 and u_2 both solve

$$\begin{cases} u_t - \triangle u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T]. \end{cases}$$

If

$$u_1(x,T) = u_2(x,T) \quad (x \in U),$$

then

$$u_1 \equiv u_2 \text{ in } U_T.$$

Proof: Again write $w = u_1 - u_2$ and set

$$e(t) = \int_U w^2(x,t)dx.$$

As before,

$$e'(t) = -2\int_U |\nabla w|^2 dx.$$

Furthermore,

$$e''(t) = -4\int_U \nabla w \cdot \nabla w_t dx = 4\int_U w_t \triangle w dx = 4\int_U (\triangle w)^2 dx.$$

By integration by parts and then Cauchy-Schwarz,

$$e'(t) = 2\int_U w\triangle w dx \leq 2\left(\int_U w^2 dx\right)^{1/2}\left(\int_U (\triangle w)^2 dx\right)^{1/2}.$$

Thus,

$$e'(t)^2 \leq e(t)e''(t).$$

This implies that

$$(\log(e(t)))'' = \left(\frac{e'(t)}{e(t)}\right)' = \frac{e(t)e''(t) - e'(t)^2}{e(t)^2} \geq 0.$$

so if $f(t) = \log(e(t))$, then f is *convex* and thus

$$f(t) \geq f(0) + tf'(0)$$

i.e.

$$\log(e(t)) \geq \log(e(0)) - at$$

where $a > 0$ because $f'(t) < 0$. Then

$$e(t) \geq e(0)e^{-at}$$

and so $e(t)$ *cannot decay faster than exponentially.* Thus if $e(T) = 0$ for some finite T, it must be that $e(t) \equiv 0$ for all $t \in [0, T]$.

2.4 Wave Equation.

In this section we study the wave equation

$$u_{tt} - \triangle u = 0 \tag{17}$$

and the nohomogeneous wave equation

$$u_{tt} - \triangle u = f \tag{18}$$

subject to general initial and boundary conditions.

Physical interpretation. The wave equation models a vibrating string (n=1), membrane (n=2), or elastic solid (n=3), and $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$.

Derivation of d'Alembert's formula.

Solution of wave equation for $n = 1$.

Assume $g \in C^2(\mathbb{R}), h \in C^1(\mathbb{R})$, and define u by **d'Alembert's formula**,

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

Then $u \in C^2(\mathbb{R} \times [0, \infty))$ and u solves

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{19}$$

Derivation: first factor the PDE as

$$(\partial_t + \partial_x)(\partial_t - \partial_x) u = 0$$

and let

$$v(x,t) := (\partial_t - \partial_x) u(x,t).$$

Therefore v satisfies the transport equation and so

$$v(x,t) = a(x-t)$$

where $a(x) = v(x,0)$. Then u satisfies the *nonhomogeneous transport equation,*

$$u_t - u_x = a(x-t)$$

which from Section 2.1, letting $f(x,t) = a(x-t)$, has solution

$$u(x,t) = g(x+t) + \int_0^t a(x + (t-s) - s)ds.$$

We also have

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

so we get

$$\begin{aligned}
u(x,t) &= g(x+t) + \int_0^t h(x+(t-s)-s) - g'(x+(t-s)-s)ds \\
&= g(x+t) - \frac{1}{2}\int_{x+t}^{x-t} h(y) - g'(y)dy \\
&= g(x+t) + \frac{1}{2}\left(g(x-t) - g(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \\
&= \frac{1}{2}(g(x+t) - g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.
\end{aligned}$$

Observations:

1. Our solution u has the form $u(x,t) = F(x+t) + G(x-t)$. Conversely, any function of this form solves $u_{tt} - u_{xx} = 0$. Hence the general solution of the 1D wave equation is a sum of the general solution of $u_t - u_x = 0$ and $u_t + u_x = 0$.

2. From d'Alembert's formula, we see that if $g \in C^k$ and $h \in C^{k-1}$ then $u \in C^k$ but is not in general smoother. So unlike the heat equation, the wave equation *does not cause instantaneous smoothing.*

1D-wave equation on the half-line. Let $\mathbb{R}_+ := \{x > 0\}$. Consider the inital/boundary-value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times (0,\infty) \end{cases} \tag{20}$$

where $g(0) = h(0) = 0$. Extend u, g, h to the whole line by *odd reflection.*

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x \leq 0 \end{cases}$$

$$\tilde{g}(x,t) := \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$

$$\tilde{h}(x) := \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x \leq 0 \end{cases}$$

Then we have

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \tilde{u} = \tilde{g},\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Thus from D'Alembert's formula, we get

$$\tilde{u}(x,t) = \frac{1}{2}(\tilde{g}(x+t) - \tilde{g}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)dy$$

and plugging in our definitions of the tilde variables we obtain a solution which tells us that (if $h = 0$) the initial profile g splits into two parts moving to the right and left with speed one:

$$u(x,t) = \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy & x \geq t \\ \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{t+x} h(y)dy & 0 \leq x \leq t \end{cases}$$

Spherical means.

We develop a method for solving the wave equation by deriving easier-to-solve equations for spherical means of the solution (the average of u in space over a sphere).

Notation.

$$U(r,t;x) := \fint_{\partial B(x,r)} u(y,t)dS(y)$$

$$G(r;x) := \fint_{\partial B(x,r)} g(y)dS(y)$$

$$H(r;x) := \fint_{\partial B(x,r)} h(y)dS(y)$$

Euler-Poisson-Darboux equation.

Fix $x \in \mathbb{R}^n$, and let u satisfy the BVP for the wave equation. Then U solves

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ U = G, U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Proof: Multiply EPD equation by r^{n-1} and rewrite it as

$$r^{n-1}U_{tt} - \partial_r\left(r^{n-1}U_r\right) = 0.$$

Then compute U_r and use the same trick as in the mean-value formula proof to get the Laplacian of u.

Solution for 3D.

Let

$$\tilde{U} := rU,\ \tilde{G} := rG,\ \tilde{H} = rH.$$

Then $\tilde{U}$ solves the *1D wave equation on the half-line*. Applying the formula we found, we find for $0 \le r \le t$,

$$\tilde{U}(r,t;x) = \frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)dy.$$

We then recover our solution u as

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0^+} U(r,t;x) \\
&= \lim_{r\to 0+} \frac{\tilde{U}}{r} \\
&= \lim_{r\to 0+} \left(\frac{\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right)}{2r} + \frac{\int_{t-r}^{t+r} \tilde{H}(y)dy}{2r} \right) \\
&= \tilde{G}'(t) + \tilde{H}(t) \\
&\vdots
\end{aligned}$$

Chapter 3: Nonlinear First-order PDE

Here, we investigate general nonlinear first-order partial differential equations of the form

$$F(Du,u,x) = 0$$

with boundary condition $u = g$ on $\Gamma \subseteq \partial U$. We want to find curves in U along which the PDE reduces to an ODE (i.e. becomes integrable). Suppose such a curve is described parametrically by

$$\mathbf{x}(s) = (x^1(s), \cdots, x^n(s))$$

and define

$$z(s) := u(\mathbf{x}(s))$$

and

$$\mathbf{p}(s) := Du(\mathbf{x}(s)).$$

1. Written entry-wise, we have $p^i(s) = u_{x_i}(\mathbf{x}(s))$. Differentiate this in s:

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s))\dot{x}^j(s)$$

2. Differentiate our PDE, $F(Du,u,x) = 0$, with respect to x_i:

$$\sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z u_{x_i} + F_{x_i} = 0.$$

3. In order to simplify the above expression, we choose to set

$$\dot{x}^j(s) = F_{p_j}$$

and now the equation, evaluated at $\mathbf{x}(s)$, is

$$\dot{p}^i(s) + F_z p^i + F_{x_i} = 0$$

4. Finally, obtain an ODE for z by differentiating $z(s)$:

$$\dot{z}(s) = \sum_{j=1}^{n} u_{x_j}(\mathbf{x}(s))\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)F_{p_j}.$$

The characteristic equations.

To summarize, we have obtained the following system of ODEs for $\mathbf{x}(s), \mathbf{p}(s), z(s)$:

$$\begin{cases} \dot{\mathbf{p}}(s) = -F_z\mathbf{p}(s) - D_{\mathbf{x}}F \\ \dot{z}(s) = D_{\mathbf{p}}F \cdot \mathbf{p} \\ \dot{\mathbf{x}}(s) = D_{\mathbf{p}}F \end{cases}$$

Case 1: F linear.

If our PDE is homogeneous and linear, it has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0,$$

for $x \in U$. Written in terms of our characteristic variables, this is

$$F(\mathbf{p}, z, \mathbf{x}) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(\mathbf{x}(s)) + c(\mathbf{x}(s))z(\mathbf{x}(s)) = 0.$$

Then we have the following system of ODEs:

$$\begin{cases} \dot{\mathbf{x}}(s) = D_{\mathbf{p}}F = \mathbf{b}(\mathbf{x}(s)) \\ \dot{z}(s) = \mathbf{b}(\mathbf{x}(s)) \cdot \mathbf{p}(\mathbf{x}(s)) = -c(\mathbf{x}(s))z(\mathbf{x}(s)) \end{cases}$$

Since $\mathbf{p}$ does not appear in either of these equations, we don't need to write down its ODE to close the system.

Example 1.

$$\begin{cases} x_1u_{x_2} - x_2u_{x_1} = u & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}$$

where U is the quadrant $\{x_1 > 0, x_2 > 0\}$ and $\Gamma = \{x_1 > 0, x_2 = 0\} \subseteq \partial U$. To match this to the above form we set

$$\mathbf{b}(x) = (-x_2, x_1),\ c(x) = -1,$$

thus the characteristic ODEs are

$$\begin{cases} \dot{x_1}(s) = -x_2(s) \\ \dot{x_2}(s) = x_1(s) \\ \dot{z}(s) = z(s) \end{cases}$$

Chapter 4: Other ways to represent solutions.

4.1. Separation of variables.

This method tries to construct a solution u to a PDE as some sort of combination of functions of fewer variables. It is best understood through examples.

Scott motivated this discussion for us by first going over how one would solve an ODE of the form

$$\begin{cases} x'(t) = Ax(t) \\ x(0) = x_0 \end{cases}$$

where A is unitarily diagonalizable, i.e.

$$A = UDU^T.$$

Our solution takes the form

$$x(t) = e^{tA}x_0.$$

Since A is diagonalizable, we can write

$$e^{tA} = Ue^{tD}U^T$$

and since the eigenvectors of A span $\mathbb{R}^n$, we can write

$$x_0 = \sum_{j=1}^{n} c_j v_j$$

where v_j are the columns of U. We thus obtain

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} v_j$$

where $c_j = (x_0, v_j)$. We would like to do something similar for PDEs...

Example 1. Initial/boundary-value problem for the heat equation:

$$\begin{cases} u_t - \triangle u = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}.$$

Conjecture that there exists a solution having the form

$$u(x, t) = v(t)w(x).$$

Plugging this in we obtain

$$0 = v'(t)w(x) - v(t)\triangle w(x)$$

which is satisfied if and only if

$$\frac{v'(t)}{v(t)} = \frac{\triangle w(x)}{w(x)}$$

and this can only be true if both of these sides equal a constant that does not depend on x or t, say, μ. The LHS equation will give

$$v'(t) = \mu v(t) \implies v(t) = c_1 e^{\mu t}.$$

To determine what μ should be, we look to the RHS equation,

$$\triangle w(x) = \mu w(x).$$

Suppose λ_i is an eigenvalue of $-\triangle$, with eigenfunction $w_i(x)$, i.e.

$$\begin{cases} -\triangle w_i(x) = \lambda_i w_i(x) \\ w_i(x) = 0 & \text{on } \partial U. \end{cases}$$

Then

$$u(x,t) = c_1 e^{-\lambda_i t} w_i(x)$$

solves the heat equation. If we have a countable number of eigenvalue/eigenfunction pairs, then our solution will be a linear combination of all of them:

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{\lambda_j t} w_j(x).$$

And we hope to choose the coefficients so that $u(x,0) = g(x)$. In fact this is valid because $-\triangle$ has eigenfunctions $w_k(x) \in H_0^1(U)$ that form an orthonormal basis for $L^2(U)$ (but this is outside the scope of PDE 1).

4.3. Transform methods.

Fourier transform.

If $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform by

$$\mathcal{F}[u] = \hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)dx$$

and its inverse Fourier transform by

$$\mathcal{F}^{-1}[\hat{u}] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} \hat{u}(y)dy$$

Plancherel's theorem.

Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\mathcal{F}[u], \mathcal{F}^{-1}[u] \in L^2(\mathbb{R}^n)$ and

$$\|\mathcal{F}[u]\|_{L^2} = \|\mathcal{F}^{-1}[u]\|_{L^2} = \|u\|_{L^2}$$

Properties of the Fourier transform.

Assume $u, v \in L^2(\mathbb{R}^n)$. Then

1. $\int_{\mathbb{R}^n} u\bar{v}dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}dy$
2. $(\hat{D^\alpha u}) = (iy)^\alpha \hat{u}$ for each multi-index α such that $D^\alpha u \in L^2(\mathbb{R}^n)$
3. $(u \hat{*} v) = (2\pi)^{n/2}\hat{u}\hat{v}$

Linear constant-coefficient PDEs.

Example: fundamental solution of heat equation. Consider the IVP

$$\begin{cases} u_t - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Taking the Fourier transform in x, we obtain an ODE for $\hat{u}(y)$,

$$\begin{cases} \hat{u}_t + |y|^2\hat{u} = 0 & \text{for } t > 0 \\ \hat{u} = \hat{g} & \text{for } t = 0, \end{cases}$$

thus we have solution

$$\hat{u}(t, y) = \hat{g}e^{-|y|^2 t}$$

and inverting the Fourier transform gives

$$u(x, t) = \mathcal{F}^{-1}[\hat{u}] = \frac{1}{(2\pi)^{n/2}} g * \mathcal{F}^{-1}[e^{-|y|^2 t}].$$

We compute

$$\mathcal{F}^{-1}[e^{-|y|^2 t}] = \frac{1}{(2t)^{n/2}} e^{-|x|^2/(4t)}$$

to recover the fundamental solution to the heat IVP that we derived previously,

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

where

$$\Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}.$$