# Methods of Applied Math Notes

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## Regular and singular perturbations.

Consider a one parameter family of problems $P(\epsilon)$ and associated solutions $S(\epsilon)$. Assume that the $\epsilon = 0$ problem is solvable.

A regular perturbation is one in which $\lim_{\epsilon \to 0} S(\epsilon) = S(0)$.

A singular perturbation is one in which $\lim_{\epsilon \to 0} S(\epsilon) \neq S(0)$.

Example 1. The problem of finding the roots of

$$x^2 + 2\epsilon x - 1 = 0$$

is an example of a regular perturbation. The exact solution is

$$x^{\epsilon}_{1,2} = -\epsilon \pm \sqrt{1 + \epsilon^2}$$

and the unperturbed solution with $\epsilon = 0$ is

$$x^0_{1,2} = \pm 1.$$

To show that $x^{\epsilon} \to x^0$, we need...

**Binomial expansion.**

If $|x| << 1$,

$$(1 + x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2} x^2 + \cdots .$$

Therefore we have

$$x^{\epsilon}_{1,2} = -\epsilon \pm \left(1 + \frac{1}{2}\epsilon^2 + O(\epsilon^4)\right) \to \pm 1$$

as $\epsilon \to 0$. We could have also found the expanded $x^{\epsilon}_{1,2}$ directly from the problem by plugging in the ansatz

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

and collecting like terms of $\epsilon$. The fact that this works also characterizes a regular perturbation.

Example 2. Finding the roots of

$$\epsilon x^2 + 2x - 1$$

is an example of a singular perturbation. The $\epsilon = 0$ case is a fundamentally different problem (linear instead of quadratic). The exact solution

$$x_{1,2}^{\epsilon} = -\frac{1}{\epsilon} \pm \frac{1}{\epsilon}\sqrt{1+\epsilon} = -\frac{1}{\epsilon} \pm \frac{1}{\epsilon}\left(1 + \frac{1}{2}\epsilon + \cdots\right)$$

converges to $\frac{1}{2}$ or $-\infty$. The unperturbed problem has one solution, $x = \frac{1}{2}$. Naively trying the same thing as before, we plug in $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$ and collect like terms to find that $x \approx \frac{1}{2} - \frac{\epsilon}{8}$. This only recovers one of the roots. We can remedy this by transforming the singular perturbation into a regular one: letting $X = \epsilon x$, our problem becomes

$$X^2 + 2X - \epsilon = 0.$$

Now if we plug in the ansatz $X = X_0 + \epsilon X_1 + \cdots$ we obtain

$$X \approx \begin{cases} \epsilon/2 + \cdots \\ -2 - \epsilon/2 + \cdots \end{cases}$$

and thus we recover the exact solution's asymptotic behavior,

$$x \approx \begin{cases} \frac{1}{2} + \cdots \\ -\frac{2}{\epsilon} - \frac{1}{2} + \cdots \end{cases}$$

Example 3. This example illustrates that even for regular perturbations, you can't always assume that the power series ansatz

$$x(\epsilon) = x_0 + \epsilon x_1 + \cdots$$

will work. More generally, you need to consider

$$x(\epsilon) = x_0 + \epsilon^{\alpha} x_1 + \cdots$$

and finding $\alpha$ is part of the solution.

## Asymptotic expansions for ODES.

Moving away from power series, we consider asymptotic expansions with a finite number of terms:

$$x(\epsilon) \sim x_0 + x_1\epsilon^{\alpha} + x_2\epsilon,$$

where $\sim$ means that the two expressions are asymptotically equal. Our only requirement is that each term is a small correction to the preceding term as $\epsilon \to 0$, therefore we must have

$$0 < \alpha < 1.$$

More generally we can expand functions as

$$f(\epsilon) \sim \sum_{n=0}^{N} a_n \delta_n(\epsilon)$$

where the $\delta_n$'s are the expansion functions. The same function can have many different asymptotic expansions, e.g.,

$$e^\epsilon \sim 1 + \epsilon + \frac{\epsilon^2}{2}$$

$$e^\epsilon \sim 1 + \sin\epsilon + \frac{1}{2}\sin^2(\epsilon).$$

However, once you've picked the expansion functions, the coefficients are **unique**. Again we only require of the expansion functions that our expansion is well-ordered: each term is a small perturbation to the preceding term as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} = 0 \quad \forall n.$$

With this requirement, the coefficients are determined sequentially as follows:

$$a_0 = \lim_{\epsilon\to 0} \frac{f(\epsilon)}{\delta_0(\epsilon)}$$

$$a_1 = \lim_{\epsilon\to 0} \frac{f(\epsilon) - a_0\delta_0(\epsilon)}{\delta_1(\epsilon)}$$

$$\vdots$$

When solving ODEs, range of validity of an expansion is the interval $t \in [0, T]$ over which the expansion remains well-ordered. We say that we have a uniformly valid expansion if $T = \infty$.

Example 1. Consider the ODE,

$$\begin{cases} \dot{y} + y = \epsilon y^\alpha \\ y(0) = 1. \end{cases}$$

We plug in a first order regular perturbation,

$$y(t;\epsilon) \sim y_0(t) + \epsilon y_1(t)$$

to obtain at $O(1)$,

$$\dot{y_0} + y_0 = 0 \implies y_0(t) = e^{-t},$$

and at $O(\epsilon)$,

$$\dot{y_1} + y_1 = y_0^\alpha = e^{-\alpha t}, \qquad y_1(0) = 0$$

Duhamel's Principle

The solution $y(t)$ of a linear constant coefficient ODE of the form

$$Ly(t) = f(t)$$

is

$$y(t) = y(0)y_h(t) + \int_0^t y_h(t-s)f(s)ds,$$

where the homogeneous solution has initial condition $y_h(0) = 1$.

Explanation: for fixed $0 < s < t$, the integrand

$$g(t;s) = y_h(t-s)f(s)$$

solves the IVP

$$\begin{cases} Lg = 0 \\ g(s) = f(s), \end{cases}$$

so Duhamel's principle tells us that the forcing term can be thought of as initial data that is being added into the system at each moment.

$$\implies y_1(t) = \int_0^t e^{-(t-s)}e^{-\alpha s}ds = \frac{e^{-\alpha t} - e^{-t}}{1-\alpha}$$

Note that $y_1$ grows in time if $\alpha < 0$. This will eventually destroy the well-orderedness of the expansion. **Two ways to extend the range of validity are the WKB method and the method of multiple scales.**

## WKB Method.

The WKB method is a method for linear differential equations with varying coefficients. We illustrate it by considering a linear harmonic oscillator,

$$\ddot{\varphi} + \Omega^2\varphi = 0$$

where $\Omega^2 = g/l$, $g$ being acceleration due to gravity and $l$ being the length of the pendulum. Suppose $\Omega$ is a slowly varying function of $t$, i.e.

$$\Omega = \Omega(\tau), \quad \tau = \epsilon t.$$

Step 1: transform ODE to slow time.

$$\epsilon^2\varphi_{\tau\tau} + \Omega^2(\tau)\varphi = 0.$$

Now we see that we have a *singular perturbation.*

Step 2: seek a solution with slowly varying amplitude and regularly varying phase,

$$\varphi(\tau) = A(\tau)e^{i\theta(\tau)/\epsilon}.$$

Step 3: plug in the ansatz and collect real and imaginary parts.

$$\varphi'(\tau) = A'e^{i\theta/\epsilon} + Ai\frac{\theta'}{\epsilon}e^{i\theta/\epsilon}$$

$$\varphi''(\tau) = \left(A'' + 2A'i\frac{\theta'}{\epsilon} + Ai\frac{\theta''}{\epsilon} - A\frac{\theta'^2}{\epsilon^2}\right)e^{i\theta/\epsilon}$$

$$\implies \epsilon^2\left(A'' + 2A'i\frac{\theta'}{\epsilon} + Ai\frac{\theta''}{\epsilon} - A\frac{\theta'^2}{\epsilon^2}\right) + \Omega^2 A = 0$$

$$\implies \epsilon^2 A'' + A(\Omega^2 - \theta'^2) + i\epsilon(2A'\theta' + A\theta'') = 0$$

Imaginary part:

$$\implies 2AA'\theta' + A^2\theta'' = 0$$

$$\implies (A^2\theta')' = 0$$

$$\implies A^2\theta' = C$$

Real part: because the ODE given by the real part contains $\epsilon$, we try plugging in asymptotic expansions for $A$ and $\theta$:

$$A(\tau,\epsilon) \sim A_0(\tau) + \epsilon A_1(\tau)$$

$$\theta(\tau,\epsilon) \sim \theta_0(\tau) + \epsilon\theta_1(\tau)$$

Then we have

$$O(1): \quad A_0(\Omega^2 - \theta_0'^2) = 0 \implies \theta_0' = \pm\Omega \implies \theta_0 = \pm\int_0^\tau \Omega(s)ds$$

Combining the real and imaginary equations gives

$$A_0^2\theta_0' = \pm A_0^2\Omega = C$$

From this we see that *the amplitude is not constant*,

$$A^2 \propto \frac{1}{\Omega}.$$

Since $\Omega = g/l$, this tells us that as the length of the pendulum increases, so does the amplitude of the oscillation (recall the Pit and the Pendulum by Edgar Allan Poe). We can now write our solution to leading order as

$$\varphi(t) \sim \sqrt{\frac{\Omega(0)}{\Omega(t)}}\left(C_1 e^{i\int_0^t \Omega(\epsilon s)ds} + C_2 e^{-i\int_0^t \Omega(\epsilon s)ds}\right)$$

The energy of the oscillation is defined as

$$E = \frac{1}{2}\left(\dot{\varphi}^2 + \Omega^2\varphi^2\right)$$

and if we compute this with our $\varphi$, we find that

$$E(t) \propto \Omega(t).$$

This is the concept in physics known as **conservation of action**, where the action of this oscillator is given by

$$\frac{E}{\Omega}.$$

## Method of multiple scales.

Consider the following second order ODE,

$$\begin{cases} \ddot{\varphi} + 2\epsilon\dot{\varphi} + \omega^2\varphi = 0 \\ \varphi(0) = 0, \;\; \dot{\varphi}(0) = \omega. \end{cases}$$

If we try a regular perturbation,

$$\varphi \sim \varphi_0(t) + \epsilon\varphi_1(t),$$

we will get that our expansion becomes disordered over time due to secular growth.

**Why does secular growth arise when the forcing term is in the kernel of the operator?**

When we are solving an ODE of the form

$$Lu = f$$

where $Lf = 0$, the forcing term aligns with the natural modes of the system governed by $L$. As a result, the system is constantly being fed energy that it is not able to counterbalance at different modes, so it leads to unbounded growth.

In the method of multiple scales, we assume

$$\varphi = \varphi(t, \tau), \;\; \tau = \epsilon t.$$

Then our ODE is transformed into a PDE - as

$$\frac{d}{dt}\varphi(t,\tau) = \varphi_t + \epsilon\varphi_\tau$$

$$\frac{d^2}{dt^2}\varphi(t,\tau) = \varphi_{tt} + 2\epsilon\varphi_{t\tau} + \epsilon^2\varphi_{\tau\tau},$$

we get

$$\varphi_{tt} + \omega^2\varphi + 2\epsilon(\varphi_{t\tau} + \varphi_t) + \epsilon^2(\varphi_{\tau\tau} + 2\varphi_\tau) = 0.$$

Plug in the asymptotic expansion

$$\varphi(t,\tau) = \varphi_0(t,\tau) + \epsilon\varphi_1(t,\tau).$$

Then at $O(1)$ we obtain

$$\varphi_0 = A(\tau)\sin(\omega t + \theta(\tau))$$

with initial conditions $A(0) = 1$ and $\theta(0) = 0$. At $O(\epsilon)$ we obtain

$$\varphi_{1tt} + \omega^2\varphi_1 = -2\varphi_{0t\tau} - 2\varphi_{0t}$$

The terms on the right hand side are

$$\varphi_{0t} = \omega A(\tau)\cos(\omega t + \theta(\tau))$$

and

$$\varphi_{0t\tau} = \omega A'(\tau)\cos(\omega t + \theta(\tau)) - \omega A(\tau)\sin(\omega t + \theta(\tau))\theta'(\tau).$$

They are all resonant terms so we set their coefficients to zero:

$$A + A' = 0, \quad A\theta' = 0$$

which leads to $A = e^{-\tau}$ and $\theta = 0$. Thus, our leading order solution is

$$\varphi_0(t, \tau) = e^{-\epsilon t} \sin(\omega t).$$

Example: Kapitza's pendulum

Consider an oscillator where the support point oscillates very quickly compared to the frequency of the pendulum. This is described by the ODE,

$$\ddot{\varphi} + \left(1 + \frac{\beta}{\epsilon} \cos\left(\frac{t}{\epsilon}\right)\right) \sin\varphi = 0, \tag{1}$$

where $\beta = \frac{Al}{g}$, $A$ being the amplitude of the oscillating support point described by $a(t) = -\epsilon A \cos(t/\epsilon)$. We encode the fast time scale in a second time variable

$$\tau = \frac{t}{\epsilon}.$$

Then our ODE becomes the PDE,

$$\varphi_{tt} + \frac{2}{\epsilon}\varphi_{t\tau} + \frac{1}{\epsilon^2}\varphi_{\tau\tau} + \left(1 + \frac{\beta}{\epsilon} \cos\tau\right) \sin\varphi = 0,$$

and after plugging in the asymptotic expansion,

$$\varphi(t, \tau) \sim \varphi_0(t, \tau) + \epsilon\varphi_1(t, \tau),$$

we obtain at $O(1)$,

$$\varphi_{0\tau\tau} = 0 \implies \varphi_0 = A(t)\tau + B(t).$$

To avoid secular growth, we must have $A(t) = 0$ and therefore we get that

$$\varphi_0 = \varphi_0(t).$$

At $O(\epsilon)$, we have

$$\varphi_{1\tau\tau} = -2\varphi_{0t\tau} - \beta \cos\tau \sin\varphi_0 = -\beta \cos\tau \sin\varphi_0,$$

which we integrate twice in $\tau$ to get

$$\varphi_1 = \beta \sin\varphi_0 \cos\tau + C_1(t)\tau + C_2(t)$$

and again we set $C_1(t) = 0$ to avoid secular growth. This doesn't tell us anything else about $\varphi_0$ because there is no danger of secular growth. We must **expand further**,

$$\varphi \sim \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2.$$

At $O(\epsilon^2)$, we get

$$\begin{aligned}
\varphi_{2\tau\tau} &= -2\varphi_{1t\tau} - \beta \cos\tau \cos\varphi_0\varphi_1 - \sin\varphi_0 - \varphi_{0tt} \\
&= -2\varphi_{1t\tau} - \beta \cos\tau \cos\varphi_0(\beta \sin\varphi_0 \cos\tau + C_2(t)) - \sin\varphi_0 - \varphi_{0tt}
\end{aligned}$$

To avoid secular growth, we need to ensure that the RHS contains no terms that are linear in $\tau$ (it doesn't) or constant in $\tau$. Clearly two culprits are $\varphi_{0tt}$ and $\sin\varphi_0$. Another hidden one comes from the fact that

$$\cos^2\tau = \frac{1}{2} + \frac{\cos 2\tau}{2}.$$

Thus we set

$$\varphi_{0tt} + \sin\varphi_0 + \frac{\beta^2}{2}\cos\varphi_0 \sin\varphi_0 = 0,$$

which can be rewritten

$$\varphi_{0tt} + \left(1 + \frac{\beta^2}{2}\cos\varphi_0\right)\sin\varphi_0 = 0$$

and further as

$$\varphi_{0tt} + \frac{d}{d\varphi_0}V(\varphi_0) = 0$$

where $V$ is the potential energy function

$$V = 1 - \cos\varphi_0 + \frac{\beta^2}{4}\sin^2\varphi_0.$$

The minima of $V$ are stable equilibrium points for $\varphi_0(t)$, since if $V' = 0$ then $\varphi_{0tt} = 0$. We determine that

$$\varphi_0 = 0$$

is always a fixed point (pendulum hanging at the bottom) and if $\beta^2 > 2$ then

$$\varphi_0 = \pi$$

is a fixed point too. Meaning the pendulum stays suspended upside down!

## Matched asymptotic expansions.

A boundary layer at $x = x^*$ is a region of width $O(\epsilon)$ over which the solution makes an $O(1)$ change. When these occur, regular perturbations will fail to match both boundary conditions. The basic idea of matched asymptotic expansions is to pose different asymptotic expansions in different parts of the domain and then match them appropriately. This was invented in 1905 to compute the airflow over aircraft wings.

Example 1.

$$\begin{cases} \epsilon y'' + y' + y = 0 \\ y(0) = 0 \\ y(1) = 1/e \end{cases}$$

Step 1: Introduce an inner variable to focus in on the boundary layer at $x = 0$:

$$X = \frac{x}{\epsilon}.$$

Step 2: Change the ODE for $y(x)$ to be in terms of $Y(X) = y(x)$.

We have

$$y'(x) = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\epsilon}Y'(X), \qquad y''(x) = \frac{1}{\epsilon^2}Y''(X)$$

and so our ODE becomes

$$Y'' + Y' + \epsilon Y = 0.$$

Step 3: Plug in an asymptotic expansion for $Y(X)$,

$$Y(X) \sim Y_0(X) + \epsilon Y_1(X).$$

At $O(1)$ we have

$$Y_0'' + Y_0' = 0 \implies Y_0 = C - Ae^{-X}$$

and applying boundary conditions $Y_0(0) = 0,\ Y_0(1/\epsilon) = 1/e$ gives

$$Y_0(X) = \frac{1}{e}(1 - e^{-X}).$$

At $O(\epsilon)$, we will see that our expansion becomes disordered for $X \in O(1/\epsilon)$, but since we've already fit all our parameters, there's nothing we can do about it. We overreached when we tried to make $Y_0$ satisfy both boundary conditions.

Step 4: Make $Y_0$ satisfy the left boundary condition and make the leading order term of the outer solution $y(x) \sim y_0(x) + \epsilon y_1(x)$ satisfy the right boundary condition.

$$O(1): \quad y_0' + y_0 = 0 \implies y_0(x) = C_1 e^{-x} = e^{-x}$$

Now we have

$$\begin{cases} Y_0(X) = C_2(1 - e^{-X}) \\ y_0(x) = e^{-x} \end{cases}$$

Step 5: Enforce that the outer and inner solutions match in the limit,

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x).$$

This gives $C_2 = 1$.

Step 6: Construct the composite expansion = inner + outer - overlap (the overlap is the matched limit).

$$y_c(x) = e^{-x} + 1 - e^{-x/\epsilon} - 1 = e^{-x} - e^{-\frac{x}{\epsilon}}$$

**Note:** if we had tried to place the boundary layer on the other side $X = \frac{x-1}{\epsilon}$ we would have found that $\lim_{X\to-\infty} Y_0(X) = \infty$ and so we wouldn't have been able to match it to the outer solution.

### Dominant balance.

In the previous example, our choice of scalings (both inner and outer) reduced the problem, to leading order, to ODEs containing only two terms, instead of 3. More generally, if we choose to introduced a scaled variable

$$X = (x - x_0)\epsilon^{\beta}$$

we can explore different choices of $\beta$ that will produce a dominant balance between two terms and leave the other one as negligibly small.

In many other examples, a **more general approach** to matching must be taken. The process is outlined as follows.

1. Pick a suitable intermediate variable $\eta = \frac{x}{\epsilon^\alpha}$, $0 < \alpha < 1$. This variable will be somewhere between $X = x/\epsilon$ and $x$.

2. Re-express both the inner and outer solutions in terms of $\eta$.
   In the previous example, if we choose $\alpha = 1/2$, this looks like

$$Y_0(X) = C_2(1 - e^{-X}) = C_2(1 - e^{-\eta/\sqrt{\epsilon}}), \quad y_0(x) = e^{-x} = e^{-\sqrt{\epsilon}x}$$

3. Demand that the two expansions are asymptotically equal $Y_0(\eta) \sim y_1(\eta)$.

A well-known boundary layer is the **gulf stream**, a swift ocean current originating in the Gulf of Mexico and flowing up along the eastern coast of the United States. Clockwise torque of winds over the Atlantic push fluid towards the equator. Because the fluid cannot pile up at the equator, it must shoot back upwards.

## The Fourier Transform

**The Fourier transform and its inverse.**

$$\hat{f}(k) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$f(x) = \mathcal{F}^{-1}(\hat{f}(k)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

We would like $f$ and $\hat{f}$ to live in the same function space...

**Schwartz Space.**

A function $f$ belongs to the Schwartz space, written $f \in S$, if $f \in C^\infty$ and $\forall\beta$,

$$\sup_\alpha \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} f \right| < \infty.$$

This means that $f$ and all its derivatives decay faster than any power of $x$ as $x \to \infty$.

**Claim.**

$$f \in S \implies \hat{f} \in S$$

**Properties:**

1. Linearity: $\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha\hat{f} + \beta\hat{g}$
2. $\hat{f}(0) = \int_{-\infty}^{\infty} f(x)dx$
3. Shift property: $\mathcal{F}(f(x - c)) = \hat{f}(k)e^{-ikc}$
4. Derivative: $\mathcal{F}(f'(x)) = ik\hat{f}(k)$

5. L2 invariance: inner product is the same in both spaces up to a factor of $2\pi$.

$$\int_{-\infty}^{\infty} f^* g dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^* \hat{g} dk$$

6. Convolution: $\mathcal{F}(f * g) = \hat{f}(k)\hat{g}(k)$

### Extension of the Fourier transform to distributions.

Distributions are "functions" that are defined by their integral against test functions ($C_0^\infty$). For example, the delta function $\delta(x)$ is defined via

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0).$$

Then we can consider *two distributions equal* if their action on test functions is always the same. For example, we can consider $x\delta(x) = 0$ since

$$\int_{-\infty}^{\infty} x\delta(x)\phi(x)dx = x\phi(x)\Big|_{x=0} = 0.$$

Even though $\delta(x) \notin S$, we would still like to consider its Fourier transform. If it had a Fourier transform, it would satisfy L2 invariance, therefore

$$\int_{\mathbb{R}} \delta(x)\phi(x)dx = \phi(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\delta}(k)\hat{\phi}(k)dk.$$

We also know that

$$\phi(0) = \frac{1}{2\pi} \int \hat{\phi}(k)dk.$$

Therefore it would have to be the case that

$$\hat{\delta}(k) = 1.$$

**Regularity of $f$ and decay rate of $\hat{f}$.**

- If $f$ has a slope discontinuity, $|\hat{f}(k)| \sim \frac{1}{k^2}$ as $|k| \to \infty$.
- If $f$ has a jump discontinuity, $|\hat{f}(k)| \sim \frac{1}{k}$ as $|k| \to \infty$.
- If $f = \delta$, $|\hat{f}(k)| \sim 1$ as $|k| \to \infty$.

### Fourier transforms for linear constant coefficient PDEs.

Example: Consider the linearized KdV equation,

$$\begin{cases} u_t = u_{xxx} \\ u(x,0) = f(x) \end{cases}.$$

Taking the Fourier transform, in $x$, of the equation yields an ODE for $\hat{u}(k,t)$ (or really an infinite family of ODEs for each $k$):

$$\begin{cases} \hat{u}_t = (ik)^3 \hat{u} \\ \hat{u}(k,0) = \hat{f}(k). \end{cases}$$

This has solution

$$\hat{u}(k,t) = \hat{f}(k) e^{-ik^3 t}.$$

Taking the inverse Fourier transform gives the solution,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - ik^3 t} \hat{f}(k) dk.$$

We can't solve this integral, but we would like to know its behavior as $t \to \infty$. Consider the solution along a ray $x = ct$,

$$u(ct,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(kc - k^3)} \hat{f}(k) dk.$$

**Riemann-Lebesgue lemma.**

Let $f \in L^1(\mathbb{R})$. Then

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = 0.$$

Therefore we know that as $t \to \infty$, the integral goes to zero. The central claim to the method of stationary phase is that the main contribution to the integral will come from stationary points, i.e. where

$$\frac{d}{dk}(kc - k^3) = 0 \implies k = \pm\sqrt{\frac{c}{3}}.$$

This tells us that along rays $x = ct$, the wave number

$$k = \pm\sqrt{\frac{c}{3}}$$

dominates.

## General method of stationary phase.

We consider integrals of the form

$$I(\epsilon) = \int_{-\infty}^{\infty} e^{i\frac{f(k)}{\epsilon}} g(k) dk, \tag{2}$$

where $f$ and $g$ are smooth, and $f'(k)$ has only isolated roots ($f'' \neq 0$). We seek an asymptotic expression as $\epsilon \to 0$. Consider a critical point $k_0$ such that

$$f'(k_0) = 0$$

and introduce notation $f_0 := f(k_0)$, $f_0' := f'(k_0)$, $f_0'' := f''(k_0)$. If we are in some neighborhood of $k_0$, then from Taylor series we have

$$g(k) = g(k_0 + r) \approx g_0 + g_0' r$$

$$f(k) = f(k_0 + r) \approx f_0 + f_0' r + f_0'' \frac{r^2}{2}$$

and thus

$$e^{i\frac{f(k)}{\epsilon}} \approx e^{i\frac{f_0}{\epsilon}} e^{\frac{if_0'' r^2}{2\epsilon}}.$$

We rescale by setting

$$s := \frac{r}{\sqrt{\epsilon}}$$

and thus $I$ becomes

$$I \sim e^{if_0/\epsilon}\sqrt{\epsilon}\int_{-\infty}^{\infty} e^{if_0'' s^2/2} g_0 ds.$$

Integral of a Gaussian.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

We rescale again to make $I$ look like this:

$$x := \frac{s}{\sqrt{2}}\sqrt{|f_0''|}\sqrt{\pm i}.$$

Finally, we get

$$\boxed{I(\epsilon) \sim e^{if_0/\epsilon}\sqrt{\epsilon} g_0 \frac{\sqrt{2\pi}}{\sqrt{|f_0''|}} e^{\pm i\frac{\pi}{4}}}$$

Returning to the integral we had when solving the **linearized kdv equation**,

$$\int_{-\infty}^{\infty} e^{it(kc - w(k))} \hat{A}(k) dk.$$

With $t = 1/\epsilon$, $kc - w(k) = f(k)$ and $\hat{A}(k) = g(k)$ and $k_0 = \sqrt{c/3}$, we obtain

$$u(x,t) \sim \frac{2}{\sqrt{2\pi t|w_0''|}} \mathrm{Re}\left(\hat{A}(k_0) e^{i(k_0 x - w_0 t - \mathrm{sgn} w_0'' \frac{\pi}{4})}\right)$$

where

$$k_0 = \sqrt{\frac{x}{3t}}, \quad w_0 = \left(\frac{x}{3t}\right)^{3/2}, \quad w_0'' = 6\sqrt{\frac{x}{3t}}.$$

### More on dispersive plane waves.

Consider a plane wave of the form

$$u(x,t) = Ae^{i(kx - wt)}$$

that is the solution of some PDE. If the PDE has real coefficients then...

- The complex conjugate of the plane wave is also a solution.
- The real and imaginary parts are solutions.

For dispersive waves the **phase velocity does not equal the group velocity.**

$$c_p := \frac{w}{k}, \quad c_g := \frac{dw}{dk}$$

## Group velocity.

First we'll look at some examples to get a feel for what the group velocity is.

Example 1. Superposition of two plane waves.

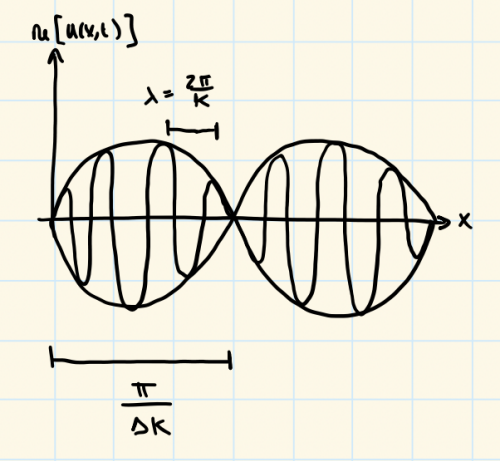
Let

$$u_1 = e^{i((k+\triangle k)x-(w+\triangle w)t)}, \quad u_2 = e^{i((k-\triangle k)x-(w-\triangle w)t)}$$

and consider

$$u = \frac{1}{2}(u_1 + u_2) = e^{i(kx-wt)}\cos(\triangle kx - \triangle wt).$$

We call the first part the **carrier wave** and the second part the **amplitude modulation**. For fixed $t$, the amplitude oscillates in space and for fixed $x$, the amplitude oscillates in time.




Along the lines in the $xt$-plane given by

$$x = \frac{\triangle w}{\triangle k}t$$

the amplitude is constant. But

$$\frac{\triangle w}{\triangle k} \to w'(k) = c_g$$

as $\triangle k \to 0$, therefore if you are moving at the group velocity, only the oscillations of the carrier wave are perceived. There is no amplitude modulation.

Example 2. Group velocity is also the speed of energy propagation.

$$\vdots$$

### Multiple branches of the dispersion relation.

Sometimes a dispersion relation will have multiple branches. For example, the PDE $u_{tt} = u_{xx}$ has dispersion relation $w = \pm k$. In general, $w_1$ and $w_2$ may be very different, and the solution to the PDE, after inverting the Fourier transform, will be

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\hat{A}_1(k)e^{-iw_1t} + \hat{A}_2(k)e^{-iw_2t}\right)e^{ikx}dk$$

where $\hat{A}_{1,2}$ are determined by the initial data. Observe that

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\hat{A}_1 + \hat{A}_2)e^{ikx}dk$$

therefore $\hat{A}_1 + \hat{A}_2 = \hat{u}(k,0)$. A similar calculation of $u_t(x,0)$ shows that

$$\begin{bmatrix} 1 & 1 \\ -iw_1 & -iw_2 \end{bmatrix}\begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} \hat{u}(k,0) \\ \hat{u}_t(k,0) \end{bmatrix}$$

so there will always be a solution as long as $w_1 \neq w_2$ i.e. we have **distinct roots**.

## Radiation condition.

Motivating example: Consider the wave equation on the half-line $x \geq 0$ with boundary condition,

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = e^{-iw_0 t}. \end{cases}$$

We already found the dispersion relation of this $w = \pm k \implies k = \pm w$, so we have a candidate solution of the form

$$u(x,t) = \left(Ae^{iw_0 x} + Be^{-iw_0 x}\right)e^{-iw_0 t}$$

and the boundary condition tells us that $A + B = 1$. But we need more conditions to uniquely determine our solution. The radiation condition gives us a second condition to make the solution unique.

- Physical POV: we construct a causal solution based on the assumption that $u(x,t)$ is caused by a wavemaker at $x = 0$ and otherwise the solution would be zero.
- Mathematical POV: the radiation condition makes the solution well-posed as $x \to \infty$ with regard to small changes in the boundary conditions.

We replace our boundary condition with a causal boundary condition,

$$\boxed{u(0,t) = e^{-iw_0 t}e^{\epsilon t}}.$$

We can now assert that

$$\boxed{\lim_{x\to\infty} u(x,t) = 0}$$

because if we started with 0 oscillations then none of the waves would reach $\infty$ in finite time. Now our solution is given by

$$\begin{aligned} u(x,t) &= \left(Ae^{iwx} + Be^{-iwx}\right)e^{-i(w_0 + i\epsilon)t} \\ &= \left(Ae^{iw_0 x - \epsilon x} + Be^{-iw_0 x + \epsilon x}\right)e^{-iw_0 t + \epsilon t} \end{aligned}$$

Since our solution must decay as $x \to \infty$, we must get $B = 0$. From our causal boundary condition we then get $A = 1$.

## Multidimensional waves.

Notation.

We now consider two dimensional waves.

$$\mathbf{x} = (x, y),\ \mathbf{k} = (k, l)$$

Plane wave:

$$Ae^{i\theta} = Ae^{i(kx+ly-wt)}.$$

We call $\theta$ the phase. Note:

$$\mathbf{k} = \nabla\theta,\ w = -\theta_t.$$

The dispersion relation $D(k, l, w) = 0 \implies w = w(k, l)$ is obtained by plugging the plane wave solution into the PDE. Wavelength:

$$\lambda = \frac{2\pi}{|\mathbf{k}|}$$

If we want to move with the phase, i.e. keep the phase constant, we set

$$0 = \frac{d\theta}{dt} = k\frac{dx}{dt} + l\frac{dy}{dt} - w$$

and we define the phase velocity $\mathbf{c}_p$ to be the speed that satisfies this equation, i.e.

$$\mathbf{k} \cdot \mathbf{c}_p = w.$$

We choose

$$\boxed{\mathbf{c}_p = \frac{w}{|\mathbf{k}|^2}\mathbf{k}}$$

to satisfy this. The group velocity in 2D follows more simply from the 1D definition,

$$\mathbf{c}_g = \nabla_{\mathbf{k}} w.$$

### Geometric Wave Theory (Ray Tracing)

Consider the variable-coefficient wave equation,

$$u_{tt} = c^2(x, y)(u_{xx} + u_{yy}),\ \ c(x, y) > 0.$$

If we try to plug in a plane-wave ansatz, we will find that $\omega = \omega(k, l, x, y)$ which means it is not a plane-wave solution. We can still make progress if $\nabla c$ is very small compared to the wavelength of the waves. We begin by restricting ourselves to time-periodic solutions:

$$u(x, y, t) = e^{-i\omega_0 t}\hat{u}(x, y).$$

Plugging this in, we obtain the **reduced wave equation,**

$$\hat{u}_{xx} + \hat{u}_{yy} + \frac{\omega_0^2}{c^2}\hat{u} = 0.$$

We introduce two new variables,

1. index of refraction: $n(x,y) = \frac{c_0}{c(x,y)}$, $c_0$ is just a suitable reference velocity that makes $n$ non-dimensional.
2. reference wave number: $\kappa_0 = \frac{\omega_0}{c_0}$

and thus obtain the reduced wave equation in its final form,

$$\hat{u}_{xx} + \hat{u}_{yy} + n^2\kappa_0^2\hat{u} = 0.$$

We say that our medium is slowly varying if

$$\frac{|\nabla c|}{c}\frac{1}{\kappa_0} << 1.$$

This suggests asymptotics based on $\frac{1}{\kappa_0} = \epsilon << 1$, referred to as high wavenumber asymptotics. Assume our solution $\hat{u}$ is of the form

$$\hat{u}(x,y) = A(x,y)e^{i\kappa_0 s(x,y)}.$$

We refer to $x$ and $y$ as slow coordinates, and $s(x,y)$ is called the **eikonal**. If we plug this in, collect real and imaginary parts, and perform some asymptotics, we obtain the following system for $A$ and $s$,

$$\boxed{\begin{cases} |\nabla s|^2 = n^2 \\ \nabla \cdot (A^2 \nabla s) = 0 \end{cases}}.$$

### Nonlinear method of characteristics.

Consider a first order PDE (in two dimensions),

$$F(x, y, u, u_x, u_y) = 0$$

with some boundary condition. The **characteristics** are lines $(x(\tau), y(\tau))$ along which the PDE reduces to an ODE. Let

$$\mathbf{x}(\tau) = (x(\tau), y(\tau))$$
$$z(\tau) = u(\mathbf{x}(\tau))$$
$$\mathbf{p}(\tau) = \nabla u(\mathbf{x}(\tau))$$

Entrywise, we have

$$p^i(\tau) = u_{x_i}(\mathbf{x}(\tau)).$$

Taking the derivative with respect to $\tau$ gives

$$\dot{p}^i(\tau) = \sum_{j=1}^{2} u_{x_i x_j}\dot{x}_j(\tau).$$

Take the derivative of the PDE with respect to $x_i$:

$$F_{x_i} + F_z p^i + \sum_{j=1}^{2} F_{p^j} u_{x_i x_j} = 0.$$

Thus we choose to set

$$\boxed{\dot{\mathbf{x}}(\tau) = \nabla_{\mathbf{p}} F(\mathbf{x}(\tau), z(\tau), \mathbf{p}(\tau))}$$

and we obtain an ODE for $\mathbf{p}(\tau)$ as well,

$$\boxed{\dot{\mathbf{p}}(\tau) = -\nabla_{\mathbf{x}} F - F_z \mathbf{p}.}$$

Finally, we obtain an ODE for $z(\tau)$ by differentiating its definition,

$$\boxed{\dot{z}(\tau) = \nabla_{\mathbf{x}} u(\mathbf{x}(\tau)) \cdot \nabla_{\mathbf{p}} F(\cdots) = \mathbf{p} \cdot \nabla_{\mathbf{p}} F.}$$

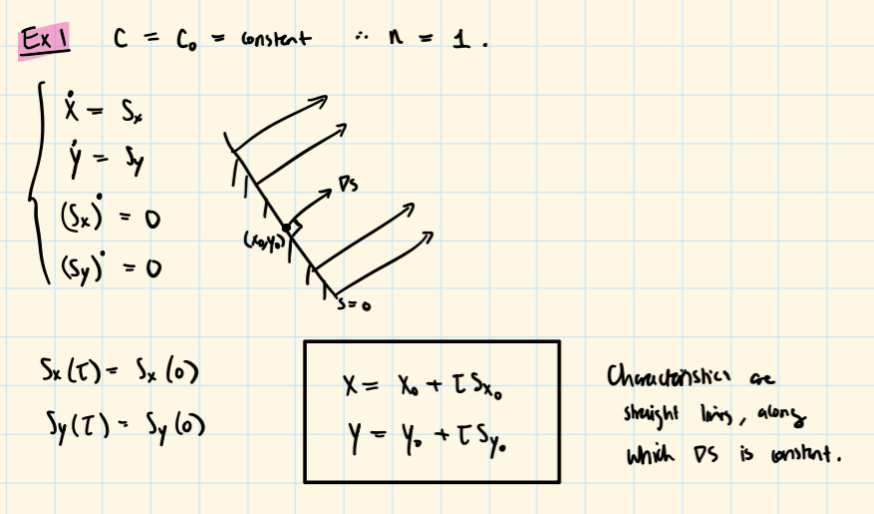
Example: Let's return to the eikonal equation, now written as

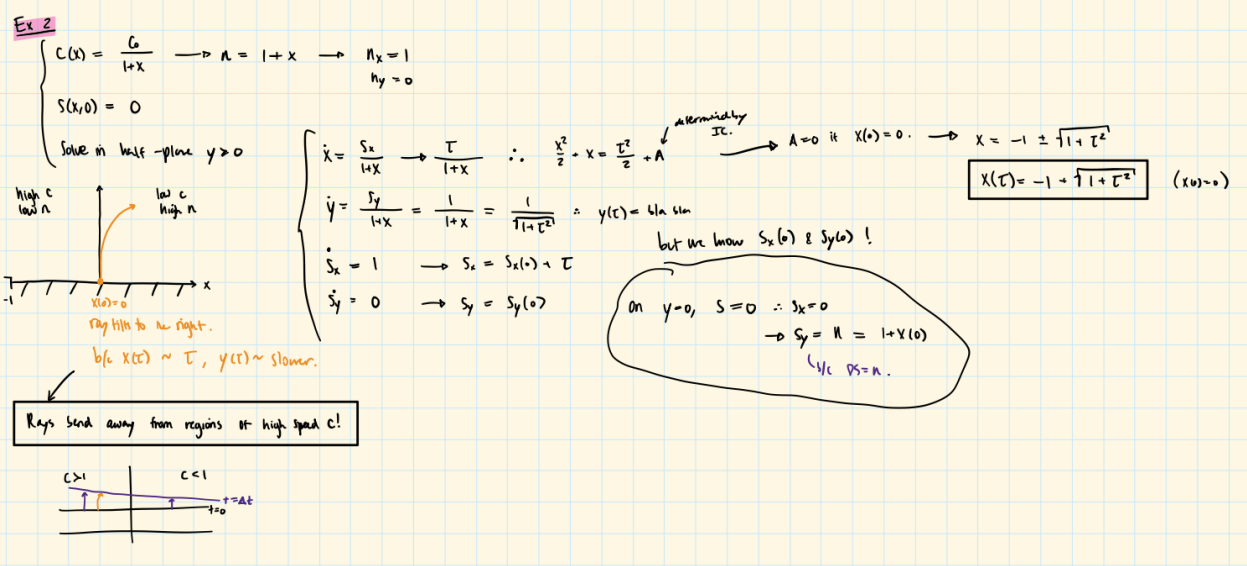
$$F = |\nabla s| - n = \sqrt{s_x^2 + s_y^2} - n(x, y) = 0.$$

This is actually a **Hamilton-Jacobi equation** since the solution itself does not appear in the PDE. The characteristics are given by

$$\dot{x}(\tau) = \frac{s_x}{n}, \; \dot{y}(\tau) = \frac{s_y}{n}$$

which tells us that we are moving normal to lines of constant $s$ (gradient points normal to level sets).






Both $\tau$ and $s(\tau)$ have direct geometric interpretations.

1. $\tau$ measures the arc length along the characteristic:

$$\sqrt{dx^2 + dy^2} = d\tau\sqrt{\dot{x}^2 + \dot{y}^2} = d\tau\sqrt{\frac{s_x^2 + s_y^2}{n^2}} = d\tau$$

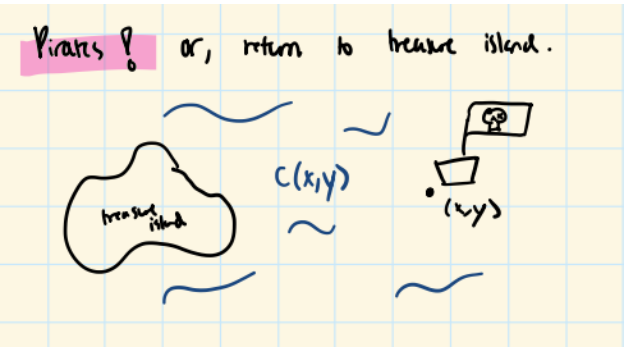
2. $s(\tau)$ measures the total travel time along the characteristic:

$$ds = s_x dx + s_y dy = d\tau(s_x\dot{x} + s_y\dot{y}) = d\tau\left(\frac{s_x^2 + s_y^2}{n}\right) = nd\tau = \frac{c_0}{c}d\tau \propto \frac{d\tau}{c}$$

   which is distance/velocity=time.

We have seen that if $c(x, y)$ is not constant, then rays will bend. Clearly they are not taking the shortest path (in terms of distance), but we will see that they are taking the path of shortest travel time. **Light chooses the path of shortest travel time!**

### Bellmann dynamic programming method.




Our goal is to travel from $(x, y)$ to the island in the shortest possible time. This could be done as a calculus of variations problem, but Bellmann dynamic programming method is easier. Assume there

is a function $T(x, y)$ that gives the shortest travel time. We want to find a PDE for $T$. Consider taking a single infinitesimally small step $(dx, dy)$. It follows from "pure thought" that

$$\boxed{T(x,y) = dt + \min\left(T(x+dx, y+dy) \quad | \quad \frac{\sqrt{dx^2+dy^2}}{c} = dt\right)}$$

This is essentially saying that the minimum amount of time it takes to get from $(x, y)$ to the island is equal to the time it took to take one step plus the minimum amount of time it takes to get from this new location to the island. In other words, this step we took was optimal in that it minimizes the remaining amount of time left in our journey. We Taylor expand

$$T(x+dx, y+dy) = T(x,y) + T_x(x+dx, y+dy)dx + T_y(x+dx, y+dy)dy + \cdots$$

and plug this in to get

$$\begin{aligned} T(x,y) &= dt + \min_{|d\mathbf{x}|=cdt} \left(T(x,y) + \nabla T \cdot d\mathbf{x}\right) \\ \implies 0 &= dt + \min_{|d\mathbf{x}|=cdt} \left(\nabla T \cdot d\mathbf{x}\right). \end{aligned}$$

The quantity $\nabla T \cdot d\mathbf{x}$ is minimized when $d\mathbf{x}$ points in the direction of $-\nabla T$, i.e. when

$$d\mathbf{x} = -cdt\frac{\nabla T}{|\nabla T|}$$

thus we obtain

$$\begin{aligned} 0 &= dt - cdt|\nabla T| \\ \implies |\nabla T| &= \frac{1}{c} \end{aligned}$$

We have recovered the **eikonal equation** (with $c_0 = 1$). Thus we see that the eikonal equation does indeed minimize the travel time.

## Varadhan asymptotics.

Recall that the heat equation in $\mathbb{R}^2$,

$$u_t = D\triangle u, \quad D > 0,$$

has fundamental solution

$$u(x, y, t) = \frac{1}{4\pi Dt} e^{-|x|^2/(4Dt)},$$

which is the solution to the heat equation with initial condition

$$u(x, y, 0) = \delta(x)\delta(y).$$

Varadhan considered a more general heat equation

$$u_t = \sum_{i,j} D_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

where $D$ is an SPD matrix. We want to also find a fundamental solution with $u(x, y, 0) = \delta(x)\delta(y)$. We seek such a solution for short times,

$$t = \epsilon << 1.$$

Assume that

$$u \sim A(x, y, t)e^{-\frac{\theta^2(x,y,t)}{4t}},$$

then as $t \to 0$, only the explicit time dependence in the exponent will matter (assuming $A$ and $\theta$ are nice and regular). The $e^{-1/4t}$ is an essential singularity. Therefore, $A_t$ and $\theta_t$ are ignorable in this limit, and so is $\nabla A$. Therefore, ignoring the time dependence of $\theta$ and $A$ gives

$$u_t \sim A\left(\frac{\theta^2}{4t^2}\right)e^{-\theta^2/4t}$$

$$\nabla u \sim A\left(-2\theta\frac{\nabla\theta}{4t}\right)e^{-\theta^2/4t}$$

$$\nabla\nabla u \sim A\left(4\theta^2\frac{\nabla\theta\nabla\theta}{16t^2}\right)e^{-\theta^2/4t}$$

Plugging this into the PDE gives (after some work)

$$1 = \nabla\theta \cdot D \cdot \nabla\theta$$

## Dispersive ray tracing.

The theory we have been exploring thus far is called geometric wave theory or ray tracing. We can actually extend this to dispersive waves, and it ends up being much simpler. Assume

$$u(\mathbf{x}, t) \sim A(\mathbf{x}, t)e^{i\theta(\mathbf{x},t)}.$$

We omit the $1/\epsilon$ factor, but assume that the phase $\theta$ is rapidly varying and the amplitude $A$ is slowly varying. As previously, we define a local wavemaker vector

$$\mathbf{k}(\mathbf{x}, t) = \nabla\theta$$

and a local frequency

$$\omega(\mathbf{x}, t) = -\theta_t.$$

Note that this is consistent with the special case from before where $\theta = \mathbf{k} \cdot \mathbf{x} - \omega t$ but now we have generalized this. Immediately, we have two important consequences from these definitions:

1. $\nabla \times \mathbf{k} = \nabla \times \nabla\theta = 0$
2. $\mathbf{k}_t + \nabla\omega = 0$

We can extract a **local dispersion relation** from the underlying PDE by freezing all coefficients and substituting in a plane wave.

Example. Consider the wave equation with variable coefficients

$$u_{tt} = c^2(\mathbf{x})\triangle u.$$

We "freeze" $c(\mathbf{x})$ and treat it as a constant. Why can we do this? Because we are assuming that the wavelengths are so small that they don't see the change in $c$, just its value at that point. We plug in a plane wave

$$u(\mathbf{x}, t) = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

to obtain a local dispersion relation:

$$\begin{aligned} -\omega^2 &= -c^2|\mathbf{k}|^2 \\ \implies D(\mathbf{k}, \omega) &= \omega^2 - c^2|\mathbf{k}|^2 = 0 \\ \implies \omega &= \Omega(\mathbf{k}, \mathbf{x}, t). \end{aligned}$$

We know that our solution is not actually a plane wave because this dispersion relation depends on space and time. Switching to component notation, we write $\mathbf{k} = (k_1, k_2, k_3)$. If we enforce consequence (2), we get the equation

$$\frac{\partial k_i}{\partial t} + \frac{\partial}{\partial x_i}\Omega(\mathbf{k}, \mathbf{x}, t) = 0$$

$$\implies \frac{\partial k_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial \Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} + \frac{\partial \Omega}{\partial x_i} = 0$$

Using consequence (1), we have that $\nabla \times \mathbf{k} = 0 \implies \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}$. Thus we obtain

$$\frac{\partial k_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial \Omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = -\frac{\partial \Omega}{\partial x_i}$$

and we can rewrite the LHS as a single operator acting on $k_i$:

$$\left(\frac{\partial}{\partial t} + \sum_{j=1}^{3} \frac{\partial \Omega}{\partial k_j}\frac{\partial}{\partial x_j}\right) k_i = -\frac{\partial \Omega}{\partial x_i}.$$

Note that $\frac{\partial \Omega}{\partial k_j}$ is the $j$th component of the group velocity. Thus we write the whole thing in vector form as

$$\left(\frac{\partial}{\partial t} + \mathbf{c_g} \cdot \nabla\right) \mathbf{k} = -\nabla\Omega.$$

The LHS is the time derivative along a group velocity ray! This means that if $\Omega$ does not depend on $\mathbf{x}$, then $\mathbf{k}$ is constant along group velocity rays. The above PDE is equivalent to a family of joint ODEs:

$$\boxed{\begin{cases} \frac{d}{dt}\mathbf{x} = \mathbf{c_g} = \frac{\partial \Omega}{\partial \mathbf{k}} \\ \frac{d}{dt}\mathbf{k} = -\frac{\partial \Omega}{\partial \mathbf{x}} \end{cases}}$$

This is a system of **Hamiltonian** ODEs for the conjugate variables $(\mathbf{x}, \mathbf{k})$ and the Hamiltonian function $\Omega(\mathbf{x}, \mathbf{k}, t)$. It is also straightforward to check that these are the characteristics of the PDE,

$$\theta_t + \Omega(\mathbf{x}, \nabla\theta, t) = 0$$

Summary of dispersive ray tracing:

1. Treat all variable coefficients as constants and extract a local dispersion relation

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$$

2. Assume a solution of the form

$$u(\mathbf{x}, t) \sim A(\mathbf{x}, t) e^{i\theta(\mathbf{x}, t)}$$

where $\mathbf{k} = \nabla\theta$ and $\omega = -\theta_t$.

3. Demand that

$$\omega(\mathbf{x}, t) = \Omega(\mathbf{k}(\mathbf{x}, t), \mathbf{x}, t)$$

and obtain the Hamiltonian system

$$\begin{cases} \frac{d}{dt}\mathbf{x} = \mathbf{c_g} = \frac{\partial \Omega}{\partial \mathbf{k}} \\ \frac{d}{dt}\mathbf{k} = -\frac{\partial \Omega}{\partial \mathbf{x}} \end{cases}$$

Example. Consider the variable-coefficient PDE

$$u_t = \alpha(x) u_{xxx}, \quad \alpha(x) > 0.$$

Following the steps outlined above, we begin by extracting a local dispersion relation:

$$-i\omega = \alpha(x)(ik)^3 \implies \omega = \Omega(k, x) = \alpha(x)k^3$$

Then our system from step 3 becomes

$$\begin{cases} \frac{dx}{dt} = 3\alpha(x)k^2 \\ \frac{dk}{dt} = -\alpha'(x)k^3 \end{cases}$$

This set of ODEs can easily be solved by a computer! Furthermore, recall that these are the characteristics of a PDE for $\theta(x, t)$ so along these lines $\theta$ solves an ODE as well, and thus we can recover the $\theta$ in $u \sim Ae^{i\theta}$.

**General fact about Hamiltonian systems.**

If $\Omega$ does not depend on time, then $\omega$ is constant along the ray:

$$\begin{aligned} \dot{\omega} &= \frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial k}\dot{k} + \frac{\partial \Omega}{\partial x}\dot{x} \\ &= -\frac{\partial \Omega}{\partial k}\frac{\partial \Omega}{\partial x} + \frac{\partial \Omega}{\partial x}\frac{\partial \Omega}{\partial k} \\ &= 0. \end{aligned}$$

# Reduction methods.

## Dimensional analysis.

Example 0: Dimensional analysis can be done to help find the form of an equation describing a physical process. Consider a pendulum of length $l$ and mass $m$. We want to describe the period $T$ of oscillations as some function of the other variables,

$$T = f(m, l, g)$$

where $g$ is acceleration due to gravity.

**Physical axiom.**

Quantities with different dimensional units can be multiplied but not added or subtracted.

By inspection of the units, $m$ cannot play a role in $f$ (because there is nothing that would cancel out its units and get back to units of time). By further inspection, to get rid of the length unit, we must divide $l$ by $g$:

$$\frac{l}{g} = \text{time}^2.$$

Therefore, we have $T = f(l/g)$. Further, to get units of just time, we conclude that $T$ must be of the form

$$T = C\sqrt{\frac{l}{g}}.$$

Now just one experiment would allow us to determine the constant $C$!

Example 0 b: Nonlinear oscillations do not follow this law. Consider again the pendulum, but now with more information about the initial angle and angular velocity,

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = s.$$

Now we need to consider

$$T = f(l, g, s).$$

The units of $s$ are 1/time (angles have no units). Again, as before we combine variables to achieve units of time:

$$T = f\left(\frac{l}{g}, s\right) = \sqrt{\frac{l}{g}}\tilde{f}\left(\frac{l}{g}, s\right).$$

**Buckingham's Π theorem.**

Any non-dimensional quantity can only depend on other non-dimensional quantities.

To apply this, we nondimensionalize the RHS by dividing by $\sqrt{l/g}$:

$$\frac{T}{\sqrt{l/g}} = \tilde{f}\left(l/g, s\right).$$

Because $l/g$ has units of time$^2$ and $s$ has units of 1/time, we can combine these to create a nondimensional quantity, $s^2 l/g$:

$$\frac{T}{\sqrt{l/g}} = \tilde{f}\left(\frac{s^2 l}{g}\right).$$

As $s \to 0$ we should recover our linear result, therefore $\tilde{f}(0) = 2\pi$. We have successfully reduced our problem from finding a function of four variables to a function of one variable.

| General fact. |
| --- |
| Dimensional reduction = number of independent units in the problem |

For Example 0, there were 3 units (length, mass, and time) and 3 parameters $(m, l, g)$ therefore there were $3 - 3 = 0$ degrees of freedom after dimensional reduction. In Example 0 b, there were the same 3 units and so the original dimension of the problem, four, was reduced to one degree of freedom.

Example 1. Consider the IVP,

$$\begin{cases} u_t + cu_x = \nu u_{xx} \\ u(x,0) = Af\left(\frac{x}{L}\right), \end{cases}$$

where $f(\cdot)$ is non-dimensional. The dimensional parameters are $A$, $L$, $\nu$, and $c$. Since $f$ is non-dimensional, $A$ must have the same units as $u$. Because it only makes sense to add things with the same units and $u_t$ has units of dimension of $u$ over time, $c$ must have units of length/time. Then $\nu$ must have units of length$^2$/time.

Key step: make the PDE non-dimensional by using the intrinsic units offered by the parameters.

$$x \to \tilde{x} = \frac{x}{L}$$
$$t \to \tilde{t} = \frac{tc}{L}$$
$$u \to \tilde{u} = \frac{u}{A}$$

Then the PDE becomes

$$\begin{cases} \tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}} = \frac{\nu}{cL}\tilde{u}_{\tilde{x}\tilde{x}} \\ \tilde{u}(\tilde{x}, 0) = f(\tilde{x}). \end{cases}$$

We have gone from having four parameters in our problem to only one. All choices of $c, L, \nu, A$ that lead to the same value of $\nu/(cL)$ are called **dynamically similar**.

### Scaling.

Previously, we were using dimensions to find scaling symmetries, but we can do this directly with the **abc-method**.

Example 1. Again, consider the non-dimensionalized Burgers equation,

$$u_t + uu_x = \nu u_{xx}.$$

We rescale as follows:

$$\begin{cases} u \to \tilde{u} = u/a \\ x \to \tilde{x} = x/b \\ t \to \tilde{t} = t/c \end{cases}$$

Then our PDE becomes

$$\frac{a}{c}\tilde{u}_{\tilde{t}} + \frac{a^2}{b}\tilde{u}\tilde{u}_{\tilde{x}} = \nu\frac{a}{b^2}\tilde{u}_{\tilde{x}\tilde{x}}$$

We want to choose our scaling parameters so that this equation reduces to the original PDE for $u$, i.e. we want

$$\frac{a}{c} = \frac{a^2}{b} = \frac{a}{b^2}$$

which gives

$$a = 1/b, \quad c = b^2.$$

We have found a 1-parameter family of scaling symmetries: if $u(x,t)$ is a solution then so is $\tilde{u}(\tilde{x},\tilde{t}) = \frac{1}{b}u\left(\frac{x}{b}, \frac{t}{b^2}\right)$.

Example 2. Consider the ODE for $\varphi(t)$:

$$\dot{\varphi} + \alpha\varphi = 0, \quad \alpha > 0.$$

Rescale time:

$$\tilde{t} = \alpha t$$

Then the ODE becomes

$$\varphi_{\tilde{t}} + \varphi = 0.$$

We got rid of the parameter $\alpha$.

### Similarity solutions.

Example. Consider the following IVP,

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

We can't use dimensional analysis since we don't have units. Also, the initial condition has no length scale: $H(bx) = H(x), b > 0$. Let's use the **abc-method** to obtain

$$a = b/c.$$

At first we have a two-parameter group of similarity solutions, but we also need to keep the initial condition consistent, i.e.

$$a\tilde{u}(\tilde{x}, 0) = H(\tilde{x})$$

means we must set $a = 1$, so we obtain $b = c$. Thus if $u(x,t)$ is a solution then so is $u(x/b, t/b)$. For simplicity, since $b$ is arbitrary, write $u(bx, bt)$. Now for the **similarity solution step**: if $u(x,t) = u(bx, bt)$, it must be the case that

$$u(x,t) = U\left(\frac{x}{t}\right).$$

Define a **similarity variable**:

$$\xi = \frac{x}{t}.$$

Plugging $u(x,t) = U(\xi)$ into the PDE we obtain an ODE of the form

$$U'(\xi)(U - \xi) = 0$$

which is satisfied if either

$$U'(\xi) = 0 \text{ or } U = \xi.$$