1d scattering slides

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1 Introduction

A function f is in the **Schwarz space** $\mathcal{S}(\mathbb{R})$ if $f \in C^\infty(\mathbb{R})$ and

$$\sup_\alpha \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} f \right| < \infty.$$

A distribution u is **singular** at x_0 if for all $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(x_0) = 1$,

$$\mathcal{F}[\psi u] \notin \mathcal{S}(\mathbb{R}).$$

Here we say that x_0 is in the **singular support** of u.

Wave front set.

Let $x_0 \in \mathbb{R}$ and let χ_+ be smooth and satisfy

$$\chi_+(\xi) := \begin{cases} 1 & \xi > 1 \\ 0 & \xi < -1 \end{cases}$$

If there exists a ψ as described above such that $\chi_+(\xi)\mathcal{F}[\psi u] \in \mathcal{S}(\mathbb{R})$, then $(x_0, +1) \notin \mathrm{WF}(u)$. Analogously, we can define χ_- to determine whether $(x_0, -1)$ is in the wavefront set.

The Fourier transform of a smooth compactly supported function $f(x)$ is **fast decreasing:** for any N there exists a constant C_N such that

$$|\hat{f}(\xi)| \leq C_N(1 + |\xi|)^{-N}.$$

This idea can be utilized to describe the singularities of a *distribution*.

A distribution u is **singular** at x_0 if for all $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(x_0) = 1$,

$$\mathcal{F}[\psi u] \text{ is not fast decreasing.}$$

If $(x_0, \pm 1) \in \mathrm{WF}(u)$ then $\chi_\pm \mathcal{F}[\psi u]$ is not Schwarz, but we can further classify the *strength* of the singularity via the **Sobolev wave front set.**

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ consists of tempered distributions $u \in \mathcal{S}'(\mathbb{R})$ such that

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty.$$

Sobolev wave front set.

If

$$\int_{\mathbb{R}} \left(1+|\xi|^2\right)^m |\chi_{\pm}(\xi)\mathcal{F}[\psi u](\xi)|^2 \, d\xi < \infty$$

then

$$(x_0, \pm 1) \notin \mathrm{WF}^m(u).$$

In other words, $\mathrm{WF}^m(u)$ consists of singularities of u around which u is not microlocally in $H^m(\mathbb{R})$.

Example: $\delta(x)$ has a singularity at $x = 0$.

Since $\mathcal{F}[\psi(x)\delta(x)] = 1$ for all $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(0) = 1$,

$$(0, \pm 1) \in \mathrm{WF}(\delta(x)).$$

We know that

$$\int 1 \cdot \left(1+|\xi|^2\right)^m d\xi < \infty$$

if $m < -\frac{1}{2}$. Therefore $(0, \pm 1) \notin \mathrm{WF}^{-m}(\delta)$ for $m > \frac{1}{2}$ and $(0, \pm 1) \in \mathrm{WF}^{-m}(\delta)$ for $m \leq \frac{1}{2}$.
The operator

$$A : u \mapsto \mathcal{F}^{-1}\left[\chi_{\pm}(\xi)\mathcal{F}[\psi u](\xi)\right]$$

is a (classical) pseudodifferential operator of order 0 with right symbol

$$a(x, \xi) = \chi_{\pm}(\xi)\psi(x).$$

This symbol is elliptic in the co-direction $(x_0, \pm 1)$. Thus we obtain an alternate definition for the wave front set:

$\mathrm{WF}(u)$

A point $(x_0, \pm 1) \notin \mathrm{WF}(u)$ if there exists a (classical) pseudodifferential operator $A \in \psi^0(\mathbb{R})$ with $\sigma_0(A)(x_0, \pm 1) \neq 0$ such that $A[u] \in \mathcal{S}(\mathbb{R})$.

Because we are interested in solutions u to $(-\partial_x^2 - k^2 + q)u = f \in \mathcal{S}(\mathbb{R})$, we only need to consider tempered distributions $u \in \mathcal{S}(\mathbb{R}) \cap C^\infty(\mathbb{R})$, so the classical wave front set is empty:

$$\mathrm{WF}(u) = \phi.$$

Scattering wave front set.

Let $\psi_\pm(x)$ be a smooth function supported near $\pm\infty$, equal to 1 for $|x|$ sufficiently large. A point $(\pm,\kappa) \in \{\pm\infty\} \times \mathbb{R}$ is not in $\mathrm{WF}_{\mathrm{sc}}(u)$ if for some $\delta > 0$ and a function $\chi_{\kappa,\delta} \in C_c^\infty(\mathbb{R})$,

$$\chi_{\kappa,\delta}(t) := \begin{cases} 1 & |t-\kappa| < \delta \\ 0 & |t-\kappa| > 2\delta, \end{cases}$$

the tempered distribution

$$P_{\chi,\pm}u(x) = \mathcal{F}^{-1}\left[\chi_{\kappa,\delta}(\xi)\mathcal{F}\left[\psi_\pm u\right](\xi)\right] \in \mathcal{S}(\mathbb{R}).$$

The scattering wave front set tells us what oscillatory behavior remains in u as $x \to \pm\infty$. If $(+,\kappa) \in \mathrm{WF}_{\mathrm{sc}}(u)$ then u has a component that looks like $e^{i\kappa x}$ as $x \to \infty$.
For example, let us compute $\mathcal{F}\left[\psi_+ e^{i\kappa x}\right]$.

$$\begin{aligned} \mathcal{F}\left[\psi_+ e^{i\kappa x}\right] &= \lim_{\sigma\to 0^-} \int_0^\infty \psi_+(x) e^{i\kappa x} e^{-i(\xi+i\sigma)x} dx \\ &= \lim_{\sigma\to 0^-} \left[\frac{\psi_+(x) e^{i(\kappa-(\xi+i\sigma))x}}{i(\kappa-(\xi+i\sigma))}\Bigg|_0^\infty - \int_0^\infty \frac{\psi_+'(x) e^{i(\kappa-(\xi+i\sigma))x}}{i(\kappa-(\xi+i\sigma))} dx \right] \\ &= \lim_{\sigma\to 0^-} -\frac{\widehat{\psi_+'}(\kappa-(\xi+i\sigma))}{i(\kappa-(\xi+i\sigma))} \end{aligned}$$

This could be singular at $\xi = k$. We need to investigate what $\widehat{\psi_+'}(\eta)$ looks like near $\eta = 0$ to know. Observe that

$$\widehat{\psi_+'}(0) = \int_{\mathbb{R}} \psi_+'(\eta) d\eta = 1.$$

Then if we let $\varphi \in C_c^\infty(\mathbb{R})$

$$\varphi(\xi) := \begin{cases} 1 & |\xi-\kappa| < \delta \\ 0 & |\xi-\kappa| > 2\delta \end{cases}$$

we have

$$\begin{aligned} \mathcal{F}\left[\psi_+ e^{i\kappa x}\right] &= \lim_{\sigma\to 0^-} -\frac{\widehat{\psi_+'}(\kappa-(\xi+i\sigma))(1-\varphi(\xi)+\varphi(\xi))}{i(\kappa-(\xi+i\sigma))} \\ &= g(\xi) - \lim_{\sigma\to 0^-} \frac{\widehat{\psi_+'}(\kappa-(\xi+i\sigma))\varphi(\xi)}{i(\kappa-(\xi+i\sigma))} \end{aligned}$$

where g is Schwarz because the zero on the numerator is the same order as the zero in the denominator. Let us apply the distribution defined by the second term to a Schwarz class function

$f(\xi)$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\widehat{\psi'_+}(\kappa-(\xi+i\sigma))\varphi(\xi)f(\xi)}{i(\kappa-(\xi+i\sigma))}d\xi &= \int_{-\infty}^{\infty} \frac{\widehat{\psi'_+}(\kappa-(\xi+i\sigma))\varphi(\xi)\left(f(\xi)-f(\kappa)+f(\kappa)\right)}{i(\kappa-(\xi+i\sigma))}d\xi \\ &= h(\sigma)+f(\kappa)\int_{-\infty}^{\infty} \frac{\widehat{\psi'_+}(\kappa-(\xi+i\sigma))\varphi(\xi)}{i(\kappa-(\xi+i\sigma))}d\xi \\ &= h(\sigma)+f(\kappa)\int_{\kappa-2\delta}^{\kappa+2\delta} \frac{\widehat{\psi'_+}(\kappa-(\xi+i\sigma))\varphi(\xi)}{i(\kappa-(\xi+i\sigma))}d\xi\end{aligned}$$

We can evaluate this integral by closing in the upper half plane and deforming the contour around the singularity at $\xi=\kappa$, to finally obtain

$$\mathcal{F}\left[\psi_+ e^{i\kappa x}\right] = a\delta(\xi-\kappa)+b\mathrm{PV}\left(\frac{1}{\xi-\kappa}\right)+s(\xi)$$

where $s\in\mathcal{S}$.